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EXISTENCE AND DECAY OF SOLUTIONS FOR A HIGHER-ORDER VISCOELASTIC WAVE EQUATION WITH LOGARITHMIC NONLINEARITY

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Abstract. The main goal of this paper is to study for the local existence and decay estimates results for a high-order viscoelastic wave equation with logarithmic nonlinearity. We obtain several results: Firstly, by using Faedo-Galerkin method and a logaritmic Sobolev inequality, we proved local existence of solutions. Later, we proved general decay results of solutions.

1. Introduction

In this paper, we investigate the following nonlinear initial boundary value problem

$$\begin{cases} u_{tt} + [Pu_{tt} + Pu_t] + Pu + u - \int_0^t g(t-s)\, Pu\, ds + u_t = u \ln |u|^k, & x \in \Omega \times (0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \\ \frac{\partial^i}{\partial \nu^i} u(x,t) = 0, \quad (i = 1,2,...,m-1) & x \in \partial\Omega \times (0,T), \end{cases} \tag{1}$$

where $\Omega \subset R^3$ is a bounded domain with smooth boundary $\partial\Omega$, v is the unit outer normal, k is positive constant to be chosen later and $P = (-\triangle)^m$, ($m \geq 1$ and $m \in N$). The kernel g has some conditions to be specified later.

The equation with the logarithmic source term is related with many branches of physics. Cause of this is interest in it occures naturally in inflation cosmology and supersymmetric field theories, quantum mechanics, nuclear physics [6, 8, 9]. Some of them authors [3–5, 11, 15, 16] improve many results in the literature.

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When $m=2$, problem (1) becomes the following

$$|u_t|^{\rho} u_{tt} + \Delta^2 u_{tt} + \Delta^2 u - \int_0^t g(t-s)\, \Delta^2 u \, ds + u = u \ln |u|^k . \tag{2}$$

In [2], Al-Gharabli et al. investigated the local existence, global existence and stability for the problem (2).

In [13], Peyravi consider

$$u_{tt} - \Delta u + u + \int_0^t g(t-s)\, \Delta u \, ds + h(u_t)\, u_t + |u|^2 u = u \ln |u|^k , \tag{3}$$

in $\Omega \subset R^3$ with $h(s) = k_0 + k_1 |s|^{m-1}$. He studied the decay estimate and exponential growth of solutions for the problem (3).

Motivated by the above studies, we asked that what results will be obtained if one revises the Laplace operator by other high order viscoelastic term. Then, we established the local existence, and general decay estimates of the solution for problem (1).

The rest of our work is organized as follows. In section 2, we give some notations and lemmas which will be used throughout this paper. In section 3, our purpose is to get suitable conditions of the local existence the solutions of the problem. In section 4, we established the general decay of the solutions of the problem.

2. Preliminaries

In this part, we give some notations and lemmas and preliminary results in order to state the main results of this paper. We use the standart Lebesgue space $L^p(\Omega)$ and Sobolev space $H^m(\Omega)$ with their scalar products and norms. Meanwhile we define $H_0^m(\Omega) = \left\{ u \in H^m(\Omega) : \frac{\partial^i u}{\partial v^i} = 0, i = 0, 1, ...m-1 \right\}$ and introduce the following abbreviations; $\|.\| = \|.\|_{L^2(\Omega)}$, $\|.\|_p = \|.\|_{L^p(\Omega)}$ and $\|.\|_{H^m} = \|.\|_{H^m(\Omega)}$ (For detailed information about these spaces, see [1, 14]). We denote by C and C_i $(i = 1, 2, ...)$ various positive constants.

Now we give some important lemmas for proof of our theorems.

Lemma 1. *[10] (Logarithmic Sobolev Inequality). Let u be any function $u \in H_0^1(\Omega)$, $\Omega \subset R^3$ be a bounded smooth domain and $\alpha > 0$ be any number. Then,*

$$\int_\Omega \ln |u|\, u^2 dx \leq \frac{\alpha^2}{2\pi} \|\nabla u\|^2 + \ln \|u\| \, \|u\|^2 - \frac{3}{2}(1 + \ln \alpha) \|u\|^2 .$$

Corollary 2. *Let u be any function $u \in H_0^m(\Omega)$, $\Omega \subset R^3$ be a bounded smooth domain and $\alpha > 0$ be any number and where c_p is he smallest positive number satisfying*

$$\|\nabla u\|^2 \leq c_p \left\| P^{\frac{1}{2}} u \right\|^2 , \quad \forall u \in H_0^m(\Omega).$$

Then, we obtain,

$$\int_{\Omega} \ln |u| u^2 dx \leq \frac{c_p \alpha^2}{2\pi} \left\| P^{\frac{1}{2}} u \right\|^2 + \ln \|u\| \|u\|^2 - \frac{3}{2} (1 + \ln \alpha) \|u\|^2 . \tag{4}$$

Lemma 3. *[7] (Logarithmic Gronwall Inequality) Assume that $w(t)$ is nonnegative, $w(t) \in L^\infty (0, T)$, $c_0 \geq 0$, and it satisfies*

$$w(t) \leq c_0 + b \int_0^t \varphi + w(s) \ln [\varphi + w(s)] ds, \ t \in [0, T],$$

where $\varphi \geq 1$ and $b > 0$ are positive constants. Then we have

$$w(t) \leq (\varphi + c_0)^{e^{bt}} - \varphi, \ t \in [0, T]. \tag{5}$$

Lemma 4. *[2] Let $\epsilon_0 \in (0, 1)$. Then there exists $d_{\epsilon_0} > 0$ such that*

$$s |\ln s| \leq s^2 + d_{\epsilon_0} s^{1-\epsilon_0}, \ \forall s > 0. \tag{6}$$

Now, we present following assumptions:
$(A1)$ $\varphi(x) = u_0(x) \in H_0^m(\Omega)$, $\zeta(x) = u_1(x) \in H_0^m(\Omega)$.
$(A2)$ $g : R^+ \to R^+$ is a C^1 nonincreasing function satisfying

$$g(0) > 0, \ \int_0^\infty g(s) ds < \infty, \ 1 - \int_0^\infty g(s) ds = l_0 > 0, \tag{7}$$

$(A3)$ There exist a nonincreasing differentiable function $\omega : R^+ \to R^+$ such that

$$g'(t) \leq -\omega(t) g(t), \ \int_0^\infty \omega(s) ds = \infty. \tag{8}$$

$(A4)$ The constant k in (1) satisfies $0 < k < k_1$,where k_1 is the positive real number satisfying

$$e^{-\frac{4}{3}} = \sqrt{\frac{2\pi l_0}{c_p k_1}} \tag{9}$$

and c_p is defined in Corollary 2.

Lemma 5. *The energy functional $E(t)$ is decreasing with respect to t. Where*

$$\begin{aligned} E(t) &= \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \left\| P^{\frac{1}{2}} u_t \right\|^2 + \frac{1}{2} \left(1 - \int_0^t g(s) ds \right) \left\| P^{\frac{1}{2}} u \right\|^2 \\ &+ \frac{1}{2} \left(g \circ P^{\frac{1}{2}} u \right) + \frac{k+2}{4} \|u\|^2 - \frac{1}{2} \int_{\Omega} \ln |u|^k u^2 dx. \end{aligned} \tag{10}$$

Proof. We multiply both sides of (1) by u_t and then integrating over Ω, we have

$$\begin{aligned}
&\int_\Omega u_{tt}u_t dx + \int_\Omega P u_{tt}u_t dx + \int_\Omega P u u_t dx \\
&+ \int_\Omega \int_0^t g(t-s) P u u_t \, ds dx + \int_\Omega u u_t dx \\
&+ \int_\Omega P u_t u_t dx + \int_\Omega u_t u_t dx \\
= &\int_\Omega \ln |u|^k u u_t dx,
\end{aligned}$$

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\left\|P^{\frac{1}{2}}u_t\right\|^2 + \frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\left\|P^{\frac{1}{2}}u\right\|^2\right. \\
&\left.+\frac{1}{2}\left(g\circ P^{\frac{1}{2}}u\right) + \frac{k+2}{4}\|u\|^2 - \frac{1}{2}\int_\Omega \ln|u|^k u^2 dx\right] \\
= \quad & -\|u_t\|^2 - \left\|P^{\frac{1}{2}}u_t\right\|^2 + \frac{1}{2}\left[+\left(g'\circ P^{\frac{1}{2}}u\right) - g(t)\left\|P^{\frac{1}{2}}u\right\|^2\right] \le 0,
\end{aligned}$$

$$E'(t) = -\|u_t\|^2 - \left\|P^{\frac{1}{2}}u_t\right\|^2 + \frac{1}{2}\left[+\left(g'\circ P^{\frac{1}{2}}u\right) - g(t)\left\|P^{\frac{1}{2}}u\right\|^2\right] \le 0. \tag{11}$$

□

Next, we begin with defining the potential energy functional and Nehari functional on $H_0^m(\Omega)$

$$\begin{cases}
I(t) = \left(1-\int_0^t g(s)\,ds\right)\left\|P^{\frac{1}{2}}u\right\|^2 + \|u\|^2 + \left(g\circ P^{\frac{1}{2}}u\right) - k\int_\Omega \ln|u|\, u^2 dx, \\
J(t) = \frac{k}{4}\|u\|^2 + \frac{1}{2}I(t), \\
E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left\|P^{\frac{1}{2}}u_t\right\|_2^2 + J(t)
\end{cases} \tag{12}$$

where

$$\left(g\circ P^{\frac{1}{2}}u\right)(t) = \int_0^t g(t-s)\int_\Omega \left|P^{\frac{1}{2}}u(t,x) - P^{\frac{1}{2}}u(s,x)\right| dx dt.$$

3. Local existence

In this section we state and prove the local existence result for problem (1). The proof is based on Faedo-Galerkin method.

Definition 6. *A function u defined on $[0,T]$ is called a weak solution of (1) if*

$$u \in C\left([0,T);H_0^m(\Omega)\right),\ u_t \in C\left([0,T);H_0^m(\Omega),u_{tt} \in C\left[0,T\right);H_0^{-m}(\Omega)\right),$$

and u satisfies

$$\begin{aligned} & \int_\Omega u_{tt}(x,t)\,w(x)\,dx + \int_\Omega P^{\frac{1}{2}}u_{tt}(x,t)\,P^{\frac{1}{2}}w(x)\,dx \\ & + \int_\Omega P^{\frac{1}{2}}u(x,t)\,P^{\frac{1}{2}}w(x)\,dx + \int_\Omega u(x,t)\,w(x)\,dx \\ & + \int_\Omega P^{\frac{1}{2}}u_t(x,t)\,P^{\frac{1}{2}}w(x)\,dx + \int_\Omega u_t(x,t)\,w(x)\,dx \\ & + \int_\Omega \int_0^t g(t-s)\left(P^{\frac{1}{2}}u, P^{\frac{1}{2}}w\right)dx \\ = & \int_\Omega \ln|u(x,t)|^k\,u(x,t)\,w(x)\,dx, \end{aligned}$$

for $w \in H_0^m(\Omega)$.

Theorem 7. *Suppose that $(A1)-(A3)$ hold and let $(u_0,u_1) \in H_0^m(\Omega) \times H_0^m(\Omega)$. Then there exists a weak solution for (1) such that*

$$u \in L^\infty\left(0,T,H_0^m(\Omega)\right),\ u_t \in L^\infty\left(0,T,H_0^m(\Omega)\right), u_{tt} \in L^\infty(0,T,H_0^{-m}(\Omega).$$

Proof. We will use the Faedo-Galerkin method to construct approximate solutions. Let $\{w_j\}_{j=1}^\infty$ be an orthogonal basis of the "separable" space $H_0^m(\Omega)$. Let

$$V_m = \operatorname{span}\{w_1,w_2,...,w_m\},$$

and let the projections of the initial data on the finite dimensional subspace V_m be given by

$$\begin{aligned} u_0^m(x) &= \sum_{j=1}^m a_j w_j(x) \to u_0 \text{ in } H_0^m(\Omega), \\ u_1^m(x) &= \sum_{j=1}^m b_j w_j(x) \to u_1 \text{ in } H_0^m(\Omega), \end{aligned} \tag{13}$$

for $j=1,2,...,m$.

We look for the approximate solution

$$u^m(x,t)=\sum_{j=1}^{m}h_j^m(t)\,w_j(x),$$

of the approximate problem in V_m

$$\begin{cases}\int\limits_\Omega\Big[u_{tt}^m w+\ P^{\frac12}u_{tt}^m\Delta w+P^{\frac12}u^mP^{\frac12}w+u^mw\\ \quad+P^{\frac12}u_t^mP^{\frac12}w+u_t^mw+\int\limits_0^t g(t-s)\left(P^{\frac12}u^m,P^{\frac12}w\right)\Big]dx\\ =\int\limits_\Omega\ln|u^m|^k\,u^mwdx,\qquad\qquad w\in V_m,\\ u^m(0)=u_0^m=\ \sum\limits_{j=1}^m(u_0,w_j)\,w_j,\\ u_t^m(0)=u_1^m=\ \sum\limits_{j=1}^m(u_1,w_j)\,w_j.\end{cases}\tag{14}$$

This leads to a system of ordinary differantial equations for unknown functions $h_j^m(t)$. Based on standard existence theory for ordinary differantial equation, one can obtain functions

$$h_j:[0,t_m)\to R,\quad j=1,2,...,m,$$

which satisfy (14) in a maximal interval $[0,t_m)$, $0<t_m\le T$. Next, we show that $t_m=T$ and that the local solution is uniformly bounded independent of m and t. For this purpose, let us replace w by u_t^m in (14)

$$\begin{aligned}&\frac{d}{dt}\Bigg[\frac12\|u_t^m\|_2^2+\frac12\left\|P^{\frac12}u_t^m\right\|^2+\frac12\left(1-\int\limits_0^t g(s)\,ds\right)\left\|P^{\frac12}u^m\right\|^2\\&+\frac12 g\circ P^{\frac12}u^m+\frac{k+2}{4}\|u^m\|^2-\frac12\int\limits_\Omega\ln|u^m|^k(u^m)^2dx\Bigg]\\=\ &-\|u_t^m\|_2^2-\left\|P^{\frac12}u_t^m\right\|^2+\frac12\left[\left(g'\circ P^{\frac12}u^m\right)-g(t)\left\|P^{\frac12}u^m\right\|^2\right],\end{aligned}\tag{15}$$

so that we can write

$$\begin{aligned}\frac{d}{dt}E^m(t)&=-\|u_t^m\|_2^2-\left\|P^{\frac12}u_t^m\right\|^2+\frac12\left[\left(g'\circ P^{\frac12}u^m\right)-g(t)\left\|P^{\frac12}u^m\right\|^2\right]\\&\le\frac12\left(g'\circ P^{\frac12}u^m\right)\le0.\end{aligned}\tag{16}$$

Then by integrating (16) with respect to t from 0, we obtain

$$E^m(t)\le E^m(0).\tag{17}$$

If we use the Logarithmic Sobolev Inequality for estimating $\frac{1}{2}\int\limits_{\Omega}\ln|u^m|^k (u^m)^2\,dx$ term we lead to

$$\begin{aligned} E^m(t) &\geq \frac{1}{2}\|u_t^m\|^2+\frac{1}{2}\left\|P^{\frac{1}{2}}u_t^m\right\|^2 \\ &+\frac{1}{2}\left[\left(1-\int\limits_0^t g(s)\,ds\right)\left\|P^{\frac{1}{2}}(u^m)\right\|^2+\left(g\circ P^{\frac{1}{2}}u^m\right)+\frac{k+2}{2}\|u^m\|^2\right] \\ &-\frac{k}{2}\left[\frac{c_p\alpha^2}{2\pi}\left\|P^{\frac{1}{2}}u^m\right\|^2+\ln\|u^m\|\,\|u^m\|^2-\frac{3}{2}(1+\ln\alpha)\|u^m\|^2\right], \\ &= \frac{1}{2}\left(\|u_t^m\|^2+\left\|P^{\frac{1}{2}}u_t^m\right\|^2\right)+\frac{1}{2}\left(l_0-\frac{kc_p\alpha^2}{2\pi}\right)\left\|P^{\frac{1}{2}}u^m\right\|^2+\frac{1}{2}g\circ P^{\frac{1}{2}}u^m \\ &+\left(\frac{k+2}{4}+\frac{3k}{4}(1+\ln\alpha)\right)\|u^m\|^2-\frac{k}{2}\ln\|u^m\|\,\|u^m\|^2, \end{aligned} \tag{18}$$

by combining (17) and (18), we obtain

$$\begin{aligned} &\|u_t^m\|^2+\left\|P^{\frac{1}{2}}u_t^m\right\|^2+\left(l_0-\frac{kc_p\alpha^2}{2\pi}\right)\left\|P^{\frac{1}{2}}u^m\right\|^2 \\ &+g\circ P^{\frac{1}{2}}u^m+\left(\frac{k+2}{2}+\frac{3k}{2}(1+\ln\alpha)\right)\|u^m\|^2 \\ \leq\ & C+k\|u^m\|^2\ln\|u^m\|^2, \end{aligned} \tag{19}$$

where $C=2E^m(0)$.

Choosing $e^{-\frac{4}{3}-\frac{2}{3k}}<\alpha<\sqrt{\frac{2\pi l_0}{kc_p}}$ will make

$$\left(l_0-\frac{kc_p\alpha^2}{2\pi}\right)>0$$

and

$$\frac{k+2}{2}+\frac{3k}{2}(1+\ln\alpha)>0.$$

This selection is possible thanks to $(A4)$ .So, we obtain

$$\begin{aligned} &\|u_t^m\|^2+\left\|P^{\frac{1}{2}}u_t^m\right\|^2+\left\|P^{\frac{1}{2}}u^m\right\|^2+\left(g\circ P^{\frac{1}{2}}u^m\right)+\|u^m\|^2 \\ \leq\ & C\left(1+\|u^m\|^2\ln\|u^m\|^2\right). \end{aligned} \tag{20}$$

We know that

$$u^m(.,t)=u^m(.,0)+\int\limits_0^t\frac{\partial u^m}{\partial\tau}(.,\tau)\,d\tau.$$

We make use of the following Cauchy-Schwarz inequality

$$(a+b)^2\leq 2\left(a^2+b^2\right),$$

we obtain

$$\begin{aligned}\|u^m(t)\|^2 &= \left\| u^m(.,0) + \int_0^t \frac{\partial u^m}{\partial \tau}(.,\tau)\, d\tau \right\|^2 \\ &\leq 2\|u^m(0)\|^2 + 2\left\| \int_0^t \frac{\partial u^m}{\partial \tau}(.,\tau)\, d\tau \right\|^2 \\ &\leq 2\|u^m(0)\|^2 + \max\{1,2T\}\frac{1+C}{C}\int_0^t \|u_t^m(\tau)\|^2 d\tau. \qquad (21)\end{aligned}$$

Then by (20) we have

$$\|u^m\|^2 \leq M + N\int_0^t \|u^m\|^2 \ln\|u^m\|^2 d\tau, \qquad (22)$$

where

$$M = 2\|u^m(0)\|^2 + \max\{1,2T\}(1+C)T, \;\; N = \max\{1,2T\}(1+C) \geq 1.$$

Noting that $x\ln x \leq (x+B)\ln(x+B)$ for any $x>0, B\geq 1$ holds, then by using of Logarithmic Gronwall inequality, we obtain

$$\|u^m\|^2 \leq (M+N)^{e^{Nt}} - N \leq C_T. \qquad (23)$$

Hence, from inequality (23) and (20)

$$\|u_t^m\|_2^2 + \|\Delta u_t^m\|^2 + \left\|P^{\frac{1}{2}}u^m\right\|^2 + g\circ P^{\frac{1}{2}}u^m + \|u^m\|^2 \leq C_2, \qquad (24)$$

where C_2 is a positive constant independent of m and t.

So, the approximate solution is uniformly bounded independent of m and t. Therefore, we can extend t_m to T.

Substituting $w = u_{tt}^m$ in (14) and using of

$$\left|\int_\Omega u u_{tt} dx\right| \leq \delta\|u_{tt}\|^2 + \frac{1}{4\delta}\|u\|^2$$

inequality, we get

$$\begin{aligned}\|u_{tt}^m\|^2 + \left\|P^{\frac{1}{2}}u_t^m\right\|^2 &= -\int_\Omega P^{\frac{1}{2}}uP^{\frac{1}{2}}u_{tt}dx + \int_\Omega\int_0^t g(t-s)P^{\frac{1}{2}}u^mP^{\frac{1}{2}}u_{tt}^m\, dsdx \\ &\quad -\int_\Omega u^m u_{tt}^m dx + \int_\Omega \ln|u^m|^k u^m u_{tt}^m dx\end{aligned}$$

$$
\begin{aligned}
&\leq \delta_1 \left\| P^{\frac{1}{2}} u_{tt}^m \right\|^2 + \delta_2 \left\| u_{tt}^m \right\|^2 + \delta_3 \left\| P^{\frac{1}{2}} u_{tt}^m \right\| \\
&\quad + \frac{1}{4\delta_3} \left(\int_0^t g(t-s) \left\| P^{\frac{1}{2}} u^m \right\|_2 ds \right)^2 + \frac{1}{4\delta_1} \left\| P^{\frac{1}{2}} u^m \right\|^2 \\
&\quad + \frac{1}{4\delta_2} \left\| u^m \right\|^2 + k \int_\Omega \ln |u^m| \, u^m u_{tt}^m dx. \qquad (25)
\end{aligned}
$$

To estimate the last term of (25), we will use (6) with $\epsilon_0 = \frac{1}{2}$ and Young's, Cauchy-Schwarz and the Embedding inequalities, we obtain

$$
\begin{aligned}
& k \int_\Omega \ln |u^m| \, u^m u_{tt}^m dx \\
\leq\ & c \int_\Omega \left(|u^m|^2 + d_2 \sqrt{u^m} \right) u_{tt}^m dx \\
\leq\ & c \left(\delta_4 u_{tt}^m dx + \frac{1}{4\delta_4} \int_\Omega \left(|u^m|^2 + d_2 \sqrt{u^m} \right)^2 dx \right) \\
\leq\ & c\delta_4 \left\| P^{\frac{1}{2}} u_{tt}^m \right\|^2 + \frac{c}{4\delta_4} \left(\int_\Omega |u^m|^4 dx + \int_\Omega |u^m| \, dx \right) \\
\leq\ & c\delta_4 \left\| P^{\frac{1}{2}} u_{tt}^m \right\|^2 + \frac{c}{4\delta_4} \left(\left\| \Delta u^m \right\|_2^4 + \left\| u^m \right\| \right). \qquad (26)
\end{aligned}
$$

Combining (26) and (25) to have

$$
\begin{aligned}
& (1-\delta_2) \left\| u_{tt}^m \right\|^2 + (1 - c\delta_4 - \delta_3 - \delta_1) \left\| P^{\frac{1}{2}} u_{tt}^m \right\|^2 \\
\leq\ & \frac{1}{4\delta_3} \left(\int_0^t g(t-s) \left\| P^{\frac{1}{2}} u^m \right\|_2 ds \right)^2 + \frac{1}{4\delta_2} \left\| u^m \right\|^2 \\
& + \frac{1}{4\delta_1} \left\| P^{\frac{1}{2}} u^m \right\|^2 + \frac{c}{4\delta_4} \left(\left\| \Delta u^m \right\|_2^4 + \left\| u^m \right\| \right).
\end{aligned}
$$

Integrate the last inequality on $(0, T)$ and use (24) and $(A2)$ leads to

$$
\begin{aligned}
& (1-\delta_2) \int_0^T \left\| u_{tt}^m \right\|^2 dt + (1 - c\delta_4 - \delta_3 - \delta_1) \int_0^T \left\| P^{\frac{1}{2}} u_{tt}^m \right\|^2 dt \\
\leq\ & \frac{c}{\delta} \int_0^T \left[g \circ P^{\frac{1}{2}} u^m + \left\| P^{\frac{1}{2}} u^m \right\|^2 + \left\| \Delta u^m \right\|_2^4 + \left\| u^m \right\|^2 \right] dt. \qquad (27)
\end{aligned}
$$

From the last inequality, if we take $\delta = \min\{\delta_2, c\delta_4, \delta_3, \delta_1\} > 0$ small enough and using (24), we have the following, for some $C_3 > 0$ not depending m or t :

$$\int_0^T \left\| P^{\frac{1}{2}} u_{tt}^m \right\|^2 dt \leq C_3. \tag{28}$$

From (24) and (28), we obtain

$$\begin{cases} u^m, \text{ is uniformly bounded in } L^\infty(0,T;H_0^m(\Omega)), \\ u_t^m, \text{ is uniformly bounded in } L^\infty(0,T;H_0^m(\Omega)), \\ u_{tt}^m, \text{ is uniformly bounded in } L^2(0,T;H_0^m(\Omega)), \end{cases} \tag{29}$$

that there exists a subsequence of (u^m) (still denoted by (u^m)), such that

$$\begin{cases} u^m \to u, \text{ weakly* in } L^\infty(0,T;H_0^m(\Omega)), \\ u_t^m \to u_t, \text{ weakly* in } L^\infty(0,T;H_0^m(\Omega)), \\ u^m \to u, \text{ weakly in } L^2(0,T;H_0^m(\Omega)), \\ u_t^m \to u_t, \text{ weakly in } L^2(0,T;H_0^m(\Omega)), \\ u_{tt}^m \to u_{tt}, \text{ weakly in } L^2(0,T;H_0^m(\Omega)). \end{cases} \tag{30}$$

Then using (29) and Aubin–Lions' lemma, we have

$$u^m \to u, \text{ strongly in } L^2(\Omega \times (0,T))$$

which implies

$$u^m \to u, \quad \Omega \times (0,T).$$

Since the map $s \to s \ln|s|^k$ is continuous, we have the convergence

$$u^m \ln|u^m|^k \to u \ln|u|^k, \quad \Omega \times (0,T). \tag{31}$$

By the Sobolev embedding theorem $(H_0^2(\Omega) \hookrightarrow L^\infty(\Omega))$, it is clear that $\left| u^m \ln|u^m|^k - u\ln|u|^k \right|$ is bounded in $L^\infty(\Omega \times (0,T))$. Next, taking into account the Lebesgue bounded convergence theorem, we have

$$u^m \ln|u^m|^k \to u \ln|u|^k, \text{strongly in } L^2\left(0,T;L^2(\Omega)\right). \tag{32}$$

We integrate (14) over $(0,t)$ to obtain, $\forall w \in V_m$

$$\begin{aligned} &\frac{1}{2}\int_\Omega u_t^m w dx ds - \frac{1}{2}\int_\Omega u_1^m w dx + \frac{1}{2}\int_\Omega P^{\frac{1}{2}} u_t^m P^{\frac{1}{2}} w dx ds \\ &-\frac{1}{2}\int_\Omega P^{\frac{1}{2}} u_1^m P^{\frac{1}{2}} w dx \frac{1}{2}\int_\Omega u^m w dx ds - \frac{1}{2}\int_\Omega u_0^m w dx \\ &+\frac{1}{2}\int_\Omega P^{\frac{1}{2}} u^m P^{\frac{1}{2}} w dx ds - \frac{1}{2}\int_\Omega P^{\frac{1}{2}} u_0^m P^{\frac{1}{2}} w dx \end{aligned}$$

$$+\int_0^t\int_\Omega P^{\frac{1}{2}}u^m P^{\frac{1}{2}}wdxds+\int_0^t\int_\Omega u^m wdxds$$
$$+\int_0^t\int_\Omega\left(\int_0^\tau g(t-s)P^{\frac{1}{2}}u^m\right)P^{\frac{1}{2}}wdsd\tau dx$$
$$=\int_0^t\int_\Omega \ln|u^m|^k u^m wdxds. \tag{33}$$

Convergences (13), (30) ,(32) are sufficient to pass to the limit in (33) as $m\to\infty$,

$$\frac{1}{2}\int_\Omega u_t^m wdxds=\frac{1}{2}\int_\Omega u_1^m wdx-\int_\Omega P^{\frac{1}{2}}u_t P^{\frac{1}{2}}wdxds$$
$$-\frac{1}{2}\int_\Omega u^m wdxds+\frac{1}{2}\int_\Omega u_0^m wdx-\frac{1}{2}\int_\Omega P^{\frac{1}{2}}u^m P^{\frac{1}{2}}wdxds$$
$$+\frac{1}{2}\int_\Omega P^{\frac{1}{2}}u_0^m P^{\frac{1}{2}}wdx+\int_\Omega P^{\frac{1}{2}}u_1^m P^{\frac{1}{2}}wdx-\int_0^t\int_\Omega P^{\frac{1}{2}}uP^{\frac{1}{2}}wdxds$$
$$-\int_0^t\int_\Omega\left(\int_0^\tau g(t-s)P^{\frac{1}{2}}u^m\right)P^{\frac{1}{2}}wdsd\tau dx$$
$$-\int_0^t\int_\Omega u^m wdxds+\int_0^t\int_\Omega \ln|u^m|^k u^m wdxds, \tag{34}$$

which implies that (34) is valid $\forall w\in H_0^m(\Omega)$.Using the fact that the terms in the right-hand side of (34) are absolutely continuous since they are functions of t defined by integrals over $(0,t)$, hence it is differentiable for a.e. $t\in R^+$. Thus, differentiating (34), we obtain, for a.e. $t\in(0,T)$ and any $w\in H_0^m(\Omega)$,

$$\int_\Omega u_{tt}wds+\int_\Omega P^{\frac{1}{2}}u_{tt}P^{\frac{1}{2}}wds+\int_\Omega P^{\frac{1}{2}}uP^{\frac{1}{2}}wdx$$
$$+\int_\Omega P^{\frac{1}{2}}u_t P^{\frac{1}{2}}wdx+\int_\Omega u_t wdx+\int_\Omega uwdx$$
$$-\int_\Omega\left(\int_0^t g(t-s)P^{\frac{1}{2}}u(s)\right)P^{\frac{1}{2}}wdsdx$$

$$= \int_\Omega \ln |u(x,t)|^k u(x,t) w(x) dx.$$

This completed the proof. □

4. General Decay

In this section we study general decay of problem (1).

Now, we introduce the following

$$E_1 = \frac{k}{4} (\gamma^*)^2, C_0 = 1 + \frac{3k}{2} (1 + \ln \hat{a}), \ 0 < \hat{a} < \sqrt{\frac{\pi l_0}{k c_p}}, \ \gamma^* = e^{\frac{2C_0 - k}{2k}}. \tag{35}$$

For the logarithmic source, we assume that $k \geq 1$. The next lemma by Martinez plays an important role in our proof.

Lemma 8. *[12] Let $E : R^+ \to R^+$ be a nonincresing function and $\zeta : R^+ \to R^+$ be a C^2 increasing function such that $\zeta(0) = 0$ and $\lim_{t\to\infty} \zeta(t) = \infty$.Assume that there exists $c > 0$ for which*

$$\int_t^\infty \zeta'(s) E(s) ds \leq cE(t), \ \forall t \geq 0,$$

and then

$$E(t) \leq \lambda E(0) e^{w\zeta(t)},$$

for some positive constants w and λ.

Theorem 9. *Suppose that $(A1) - (A3)$ hold. Let $\|u_0(x)\|_2 < \gamma^*$ and $0 < E(0) < E_1$. Then, there exist two positive constants n and $\tilde{n}$ suh that*

$$E(t) \leq nE(0) e^{\left(-\tilde{n} \int_0^t \omega(s) ds\right)}, \tag{36}$$

holds for all $t > 0$.

To prove the our theorem we need the following lemma.

Lemma 10. *Assume that $(A1) - (A2)$ hold, $\|u_0(x)\|_2 < \gamma^*$ and $0 < E(0) < E_1$. Then $\|u(x,t)\|_2 < \gamma^*$ for all $t \in [0,T)$.*

Proof. By the definition of $E(t)$, using Logarithmic Sobolev inequality and (12), we obtain

$$\begin{aligned} E(t) \geq \ & J(t) \geq \frac{1}{2} I(t) = \frac{1}{2} \Bigg[\left(1 - \int_0^t g(s) ds\right) \left\|P^{\frac{1}{2}} u\right\|^2 \\ & + g \circ P^{\frac{1}{2}} u + \frac{1}{2} \|u\|^2 - k \int_\Omega \ln |u| u^2 dx \Bigg] \end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\left\|P^{\frac{1}{2}}u\right\|^2+\frac{1}{2}g\circ P^{\frac{1}{2}}u+\frac{1}{2}\|u\|^2 \\
&\quad -\frac{k}{2}\left(\frac{c_p\alpha^2}{2\pi}\left\|P^{\frac{1}{2}}u\right\|^2+\ln\|u\|\,\|u\|^2-\frac{3}{2}(1+\ln\alpha)\|u\|^2\right) \\
&\geq \frac{1}{2}\left(l_0-\frac{kc_p\alpha^2}{2\pi}\right)\left\|P^{\frac{1}{2}}u\right\|^2+\frac{1}{2}g\circ P^{\frac{1}{2}}u \\
&\quad +\frac{1}{2}\left(1+\frac{3k}{2}(1+\ln\alpha)-k\ln\|u\|\right)\|u\|^2. \qquad (37)
\end{aligned}$$

Let $\alpha=\hat{a}$, from (35) we get

$$E(t)\geq\frac{1}{2}C_0\gamma^2-\frac{k}{2}(\ln\gamma)\gamma^2=M(\gamma), \qquad (38)$$

where $\|u\|=\gamma$. It is clear that $\lim_{\gamma\to\infty}M(\gamma)=-\infty$. Now, if we define γ^* the max roote of $\frac{d}{d\gamma}M(\gamma)=0$, hence, by taking

$$\gamma^*=e^{\frac{2C_0-k}{2k}}. \qquad (39)$$

Thus, we can say

$$M(\gamma)=\begin{cases} >0, & 0\leq\gamma<\gamma^*, \\ =0, & \gamma=\gamma^*, \\ <0, & \gamma<\gamma^*<\infty \end{cases}.$$

Then if we write (39) in (38), we obtain

$$\begin{aligned}
\max_{0<\gamma<\infty}M(\gamma) &= \frac{1}{2}C_0(\gamma^*)^2-\frac{3k}{2}(\ln\gamma^*)(\gamma^*)^2 \\
&= M(\gamma^*) \\
&= \frac{k}{4}(\gamma^*)^2 \\
&= E_1. \qquad (40)
\end{aligned}$$

Assume that $\|u(x,t)\|_2<\gamma^*$ is false in $[0,T)$. Moreover , because of continuity of $u(t)$ and Bolzano Theorem, it comes after that there exists $0<t_0<T$ such that $\|u(x,t_0)\|_2=\gamma^*$. From (38) and (40) we can write that

$$E(t_0)\geq M(\|u(x,t_0)\|_2)=M(\gamma^*)=E_1.$$

But this is unfeasible since, by (12), $E(t)\leq E(0)<E_1$ for all $t\geq 0$. So that proof is completed. □

Remark 11. *Under assumptions of Lemma 10, $I(t)\geq 0$ for all $t\in[0,T)$. By the, definition of $I(t)$, Logarithic Sobolev Inequality and $\alpha=\hat{a}$, we obtain,*

$$
\begin{aligned}
I(t) &= \left(1-\int_0^t g(s)\,ds\right)\left\|P^{\frac{1}{2}}u\right\|^2+\|u\|^2+\frac{1}{2}g\circ P^{\frac{1}{2}}u-k\int_\Omega \ln|u|\,u^2dx \\
&\geq l_0\left\|P^{\frac{1}{2}}u\right\|^2+\|u\|^2-k\left[\frac{c_p\hat{a}^2}{2\pi}\left\|P^{\frac{1}{2}}u\right\|^2+\ln\|u\|\,\|u\|^2-\frac{3}{2}(1+\ln\alpha)\|u\|_2\right] \\
&\geq \left(l_0-\frac{kc_p\hat{a}^2}{2\pi}\right)\left\|P^{\frac{1}{2}}u\right\|^2+\left(1+\frac{3k}{2}(1+\ln\alpha)\|u\|_2-k\ln\|u\|\right)\|u\|^2 \\
&\geq \left(l_0-\frac{kc_p\hat{a}^2}{2\pi}\right)\left\|P^{\frac{1}{2}}u\right\|^2+\frac{k}{2}\|u\|^2 \\
&\geq 0. \qquad (41)
\end{aligned}
$$

Remark 12. *By Remark 11 we can say that if* $\|u(x,0)\|_2<\gamma^*$ *and* $E(0)<E_1$ *then* $J(t)\geq 0$ *and* $E(t)\geq 0$ *for all* $t\in[0,T)$. *Moreover from (12) and(41), for all* $t\in[0,T)$ *we obtain*

$$
\begin{cases}
\|u_t(t)\|^2\leq 2E(t)\leq 2E(0), \\
\left\|P^{\frac{1}{2}}u_t(t)\right\|^2\leq 2E(t)\leq 2E(0), \\
\|u(t)\|^2\leq \frac{2}{k}I(t)\leq\frac{4}{k}J(t)\leq\frac{4}{k}E(t)\leq\frac{4}{k}E(0), \\
\left\|P^{\frac{1}{2}}u(t)\right\|^2\leq\frac{2\pi}{2\pi l_0-kc_p(\hat{a})^2}I(t)\leq\frac{4\pi}{2\pi l_0-kc_p(\hat{a})^2}E(t)\leq\frac{4\pi}{2\pi l_0-kc_p(\hat{a})^2}E(0),
\end{cases} \qquad (42)
$$

which show that the solutions are bounded in time.

Now, we are in a situation to demonstrate the Theorem 10.

Proof. Firstly we multiply both sides of the (1) with $\omega(t)u$ and integrate on $[t_1,t_2]\times\Omega$, $0\leq t_1\leq t_2\leq\infty$, we get

$$
\begin{aligned}
&\int_{t_1}^{t_2}\omega(t)\int_\Omega uu_{tt}dxdt+\int_{t_1}^{t_2}\omega(t)\left(1-\int_0^t g(s)\,ds\right)\left\|P^{\frac{1}{2}}u\right\|^2dt \\
&+\int_{t_1}^{t_2}\omega(t)\int_\Omega P^{\frac{1}{2}}u\int_0^t g(t-s)\left(P^{\frac{1}{2}}u(t)-P^{\frac{1}{2}}u(s)\right)dsdxdt \\
&+\int_{t_1}^{t_2}\omega(t)\int_\Omega P^{\frac{1}{2}}uP^{\frac{1}{2}}u_tdxdt+\int_{t_1}^{t_2}\omega(t)\int_\Omega uu_tdxdt+\int_{t_1}^{t_2}\omega(t)\|u\|^2 \\
&+\int_{t_1}^{t_2}\omega(t)\int_\Omega P^{\frac{1}{2}}uP^{\frac{1}{2}}u_{tt}dxdt-\int_{t_1}^{t_2}\omega(t)\int_\Omega\left(\ln|u|^k\right)u^2dxdt \\
=\ & 0. \qquad (43)
\end{aligned}
$$

Then, from (43) and multiplying $E(t)$ with $\sigma\omega(t)$ $(\sigma>0)$ and integrating on $[t_1,t_2]$, we obtain

$$\begin{aligned}
\sigma\int_{t_1}^{t_2}\omega(t)E(t)\,dt &= -\int_{t_1}^{t_2}\omega(t)\int_\Omega uu_{tt}dxdt \\
&\quad -\int_{t_1}^{t_2}\omega(t)\int_\Omega P^{\frac{1}{2}}uP^{\frac{1}{2}}u_{tt}dxdt+\frac{\sigma}{2}\int_{t_1}^{t_2}\omega(t)\|u_t\|^2dt \\
&\quad +\left(\sigma\left(\frac{k+2}{4}\right)-1\right)\int_{t_1}^{t_2}\omega(t)\|u\|^2dt+\frac{\sigma}{2}\int_{t_1}^{t_2}\omega(t)\,g\circ P^{\frac{1}{2}}udt \\
&\quad +\left(\frac{\sigma}{2}-1\right)\int_{t_1}^{t_2}\omega(t)\left(1-\int_0^t g(s)\,ds\right)\left\|P^{\frac{1}{2}}u\right\|^2dt \qquad (44)\\
&\quad -\int_{t_1}^{t_2}\omega(t)\int_\Omega P^{\frac{1}{2}}uP^{\frac{1}{2}}u_tdxdt-\int_{t_1}^{t_2}\omega(t)\int_\Omega uu_tdxdt \\
&\quad +\frac{\sigma}{2}\int_{t_1}^{t_2}\omega(t)\left\|P^{\frac{1}{2}}u_t\right\|^2dt+k\left(1-\frac{\sigma}{2}\right)\int_{t_1}^{t_2}\omega(t)\int_\Omega u^2\ln|u|\,dxdt \\
&\quad +\int_{t_1}^{t_2}\omega(t)\int_\Omega P^{\frac{1}{2}}u\int_0^t g(t-s)\left(P^{\frac{1}{2}}u(t)-P^{\frac{1}{2}}u(s)\right)dsdxdt.
\end{aligned}$$

Next, we try to estimate terms in the right hand side of (44). For the first term we obtain

$$\begin{aligned}
-\int_{t_1}^{t_2}\omega(t)\int_\Omega uu_{tt}dxdt &= -\int_\Omega\omega(t)\,uu_tdx\,\Big|_{t_1}^{t_2}+\int_{t_1}^{t_2}\omega'(t)\int_\Omega uu_tdxdt \\
&\quad +\int_{t_1}^{t_2}\omega(t)\|u_t\|^2dt. \qquad (45)
\end{aligned}$$

For the first term in right side of (45), using (42) and Young's inequality we obtain

$$\left|-\int_\Omega\omega(t)\,uu_tdx\right|_{t_1}^{t_2} \le \sum_{i=1}^{2}\left|\omega(t)\int_\Omega uu_tdx\right|_{t=t_i}$$

$$\begin{aligned}
&\leq \sum_{i=1}^{2}\omega(t)\left(\frac{1}{2}\int_{\Omega}u^{2}(t)\,dt+\frac{1}{2}\int_{\Omega}u_t^{2}(t)\,dt\right)|_{t=t_i}\\
&\leq \left(\frac{4}{k}+2\right)\omega(t_1)E(t_1). \qquad (46)
\end{aligned}$$

Similary, we get

$$\begin{aligned}
\left|\int_{t_1}^{t_2}\omega'(t)\int_{\Omega}uu_t\,dxdt\right| &\leq -\left(\frac{2}{k}+1\right)\int_{t_1}^{t_2}\omega'(t)E(t)\,dt\\
&= -\left(\frac{2}{k}+1\right)\left(\omega(t)E(t)|_{t_1}^{t_2}-\int_{t_1}^{t_2}\omega(t)E'(t)\,dt\right)\\
&\leq \left(\frac{2}{k}+1\right)\omega(t_1)E(t_1). \qquad (47)
\end{aligned}$$

Also, we have

$$\begin{aligned}
&-\int_{t_1}^{t_2}\omega(t)\int_{\Omega}P^{\frac{1}{2}}uP^{\frac{1}{2}}u_{tt}\,dxdt\\
=&-\int_{\Omega}\omega(t)P^{\frac{1}{2}}uP^{\frac{1}{2}}u_t\,dx\,|_{t_1}^{t_2}+\int_{t_1}^{t_2}\omega'(t)\int_{\Omega}P^{\frac{1}{2}}uP^{\frac{1}{2}}u_t\,dxdt\\
&+\int_{t_1}^{t_2}\omega(t)\left\|P^{\frac{1}{2}}u_t\right\|^{2}dt. \qquad (48)
\end{aligned}$$

For the first term in right side of (48), we use (42) and Young inequality to obtain

$$\begin{aligned}
&\left|-\int_{\Omega}\omega(t)P^{\frac{1}{2}}uP^{\frac{1}{2}}u_t\,dx\right|_{t_1}^{t_2}\\
\leq&\sum_{i=1}^{2}\left|\omega(t)\int_{\Omega}\omega(t)P^{\frac{1}{2}}uP^{\frac{1}{2}}u_t\,dx\right|_{t=t_i}\\
\leq&\sum_{i=1}^{2}\omega(t)\left(\frac{1}{2}\int_{\Omega}P^{\frac{1}{2}}u^{2}(t)\,dt+\frac{1}{2}\int_{\Omega}P^{\frac{1}{2}}u_t^{2}(t)\,dt\right)|_{t=t_i}\\
\leq&\left(\frac{4\pi}{2\pi l_0-kc_p(\hat{a})^{2}}+2\right)\omega(t_1)E(t_1). \qquad (49)
\end{aligned}$$

Similary, we get

$$\begin{aligned}
&\left|\int_{t_1}^{t_2}\omega'(t)\int_\Omega P^{\frac{1}{2}}uP^{\frac{1}{2}}u_t dxdt\right| \\
\leq\ & -\left(\frac{2\pi}{2\pi l_0-kc_p(\hat{a})^2}+1\right)\int_{t_1}^{t_2}\omega'(t)E(t)\,dt \\
=\ & -\left(\frac{2\pi}{2\pi l_0-kc_p(\hat{a})^2}+1\right)\left(\omega(t)E(t)\,|_{t_1}^{t_2}-\int_{t_1}^{t_2}\omega(t)E'(t)\,dt\right) \\
\leq\ & \left(\frac{2\pi}{2\pi l_0-kc_p(\hat{a})^2}+1\right)\omega(t_1)E(t_1). \qquad (50)
\end{aligned}$$

For the estimating last term of (44) using Young's inequality, (42) and (7), for $\varepsilon>0$, we obtain

$$\begin{aligned}
&\left|\int_\Omega P^{\frac{1}{2}}u\int_0^t g(t-s)\left(P^{\frac{1}{2}}u(t)-P^{\frac{1}{2}}u(s)\right)dsdx\right| \\
\leq\ & \frac{\varepsilon}{2}\left\|P^{\frac{1}{2}}u(t)\right\|^2+\frac{1}{2\varepsilon}\int_\Omega\left|\int_0^t g(t-s)\left(P^{\frac{1}{2}}u(t)-P^{\frac{1}{2}}u(s)\right)ds\right|^2dx \\
\leq\ & \frac{\varepsilon}{2}\left\|P^{\frac{1}{2}}u(t)\right\|^2+\frac{1}{2\varepsilon}\left(\int_0^t g(s)\,ds\right)\int_\Omega\int_0^t g(t-s)\left|\left(P^{\frac{1}{2}}u(t)-P^{\frac{1}{2}}u(s)\right)\right|^2dsdx \\
\leq\ & \frac{4\pi\varepsilon}{2\pi l_0-kc_p(\hat{a})^2}E(t)+\frac{1-l_0}{2\varepsilon}\left(g\circ P^{\frac{1}{2}}u\right)(t). \qquad (51)
\end{aligned}$$

By using Young inequality for $\delta>0$ and $\delta_1>0$ and (42), we have

$$\begin{aligned}
&\left|\int_{t_1}^{t_2}\omega(t)\int_\Omega P^{\frac{1}{2}}uP^{\frac{1}{2}}u_t dxdt+\int_{t_1}^{t_2}\omega(t)\int_\Omega uu_t dxdt\right| \\
\leq\ & \frac{1}{2}\left(\delta\|u\|^2+\delta_1\left\|P^{\frac{1}{2}}u\right\|^2\right) \\
& \frac{1}{2\delta}\|u_t\|^2+\frac{1}{2\delta_1}\left\|P^{\frac{1}{2}}u_t\right\|^2 \\
\leq\ & \left(\frac{2\delta}{k}+\frac{2\pi\delta_1}{2\pi l_0-kc_p(\hat{a})^2}\right)E(t)
\end{aligned}$$

$$\frac{1}{2\delta}\left\|u_t\right\|^2+\frac{1}{2\delta_1}\left\|P^{\frac{1}{2}}u_t\right\|^2$$

Hence, by (45)- (50) and (7) the equality (44), for $\sigma < 2$, brings about to the following

$$\begin{aligned}
&\sigma\int_{t_1}^{t_2}\omega(t)\,E(t)\,dt\\
\leq\quad&\left(\frac{6}{k}+\frac{6\pi}{2\pi l_0-kc_p(\hat{a})^2}+6\right)\omega(t_1)\,E(t_1)\\
&+\left(\frac{4\pi\varepsilon}{2\pi l_0-kc_p(\hat{a})^2}+\frac{2\delta}{k}+\frac{2\pi\delta_1}{2\pi l_0-kc_p(\hat{a})^2}\right)\int_{t_1}^{t_2}\omega(t)\,E(t)\\
&+\left(\frac{\sigma}{2}+1+\frac{1}{2\delta}\right)\int_{t_1}^{t_2}\omega(t)\left\|u_t\right\|^2dt+\left(\frac{\sigma}{2}+1+\frac{1}{2\delta_1}\right)\int_{t_1}^{t_2}\omega(t)\left\|P^{\frac{1}{2}}u_t\right\|^2dt\\
&+\left(\frac{\sigma}{2}+\frac{1-l_0}{2\varepsilon}\right)\int_{t_1}^{t_2}\omega(t)\,g\circ P^{\frac{1}{2}}u\,dt+\left(\sigma\left(\frac{k+2}{4}\right)-1\right)\int_{t_1}^{t_2}\omega(t)\left\|u\right\|^2dt\\
&+\left(\frac{\sigma}{2}-1\right)\int_{t_1}^{t_2}\omega(t)\left(1-\int_0^t g(s)\,ds\right)\left\|P^{\frac{1}{2}}u\right\|^2dt\\
&+k\left(1-\frac{\sigma}{2}\right)\int_{t_1}^{t_2}\omega(t)\int_\Omega u^2\ln|u|\,dx\,dt.
\end{aligned}\tag{52}$$

Because of (8) and (11) clearly we have

$$\int_{t_1}^{t_2}\omega(t)\left\|u_t\right\|^2dt\leq\omega(t_1)\,E(t_1)\,,\quad\int_{t_1}^{t_2}\omega(t)\left\|P^{\frac{1}{2}}u_t\right\|^2dt\leq\omega(t_1)\,E(t_1)\tag{53}$$

and

$$\omega(t)\left(g\circ P^{\frac{1}{2}}u\right)(t)\leq-\left(g'\circ P^{\frac{1}{2}}u\right)(t)\leq2\omega(t_1)\,E(t_1)\,.\tag{54}$$

By using of (53) and (54), Logarithmic Sobolev inequality for $\alpha=\hat{a}$ in (52) we get

$$\begin{aligned}
&\left(\sigma-\frac{4\pi\varepsilon}{2\pi l_0-kc_p(\hat{a})^2}-\frac{2\delta}{k}-\frac{2\pi\delta_1}{2\pi l_0-kc_p(\hat{a})^2}\right)\int_{t_1}^{t_2}\omega(t)\,E(t)\,dt\\
\leq\quad&\left[\frac{6}{k}+\frac{6\pi}{2\pi l_0-kc_p(\hat{a})^2}+6+\left(\frac{\sigma}{2}+1+\frac{1}{2\delta}\right)\right.
\end{aligned}$$

$$+\left(\frac{\sigma}{2}+1+\frac{1}{2\delta_1}\right)+\sigma+\frac{1-l_0}{\varepsilon}\Bigg]\omega(t_1)E(t_1)$$
$$+\frac{1}{2}\left(\sigma\left(\frac{k+2}{2}\right)-1\right)\int_{t_1}^{t_2}\omega(t)\|u\|^2\,dt$$
$$+\left(\frac{\sigma}{2}-1\right)\left(l_0-\frac{kc_p\alpha^2}{2\pi}\right)\int_{t_1}^{t_2}\omega(t)\left\|P^{\frac{1}{2}}(u)\right\|^2dt$$
$$+\left[k\left(1-\frac{\sigma}{2}\right)\left(\ln\|u\|-\frac{3}{2}(1+\ln\alpha)\right)-\frac{1}{2}\right]\int_{t_1}^{t_2}\omega(t)\|u\|^2\,dt. \tag{55}$$

Then, we take small enough such that $0<\sigma<\frac{2}{k+2}$. For this choice of σ clearly

$$\left(\frac{\sigma}{2}-1\right)<0 \text{ and } 0<\hat{a}<\sqrt{\frac{\pi l_0}{kc_p}},$$

we get

$$\left(\frac{\sigma}{2}-1\right)\left(l_0-\frac{kc_p\alpha^2}{2\pi}\right)<0.$$

By Lemma 10 and (35) for $k\geq 1$ we also get

$$\begin{aligned}
& k\left(1-\frac{\sigma}{2}\right)\left(\ln\|u\|-\frac{3}{2}(1+\ln\alpha)\right)-\frac{1}{2}\\
< \;& k\left(1-\frac{\sigma}{2}\right)\left(\ln\gamma^*-\frac{3}{2}(1+\ln\alpha)\right)-\frac{1}{2}\\
= \;& k\left(1-\frac{\sigma}{2}\right)\left(\frac{C_0}{k}-\frac{1}{2}-\frac{3}{2}(1+\ln\alpha)\right)-\frac{1}{2}\\
= \;& k\left(1-\frac{\sigma}{2}\right)\left(\frac{1}{k}-\frac{1}{2}\right)-\frac{1}{2}\\
\leq \;& 0.
\end{aligned}$$

If we take $\varepsilon,\delta,\delta_1$ small enough so that

$$\sigma-\frac{4\pi\varepsilon}{2\pi l_0-kc_p(\hat{a})^2}-\frac{2\delta}{k}-\frac{2\pi\delta_1}{2\pi l_0-kc_p(\hat{a})^2}>0.$$

Therefore, there exist $C>0$ such that

$$\int_{t_1}^{t_2}\omega(t)E(t)\,dt\leq CE(t_1).$$

Taking $\zeta(x,0)=\int\limits_0^t \omega(s)\,ds$ and letting $t_2\to\infty$, then an application of Lemma 8 established (36). Thus, the proof is completed. □

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References

[1] Adams, R. A. and Fournier, J. J., *Sobolev spaces*, Elsevier, 2003.

[2] Al-Gharabli, M. M., Guesmia, A. and Messaoudi, S. A., Well-posedness and asymptotic stability results for a viscoelastic plate equation with a logarithmic nonlinearity, *Applicable Analysis 99* (1) (2020), 50–74.

[3] Al-Gharabli, M. M. and Messaoudi, S. A., The existence and the asymptotic behavior of a plate equation with frictional damping and a logarithmic source term, *Journal of Mathematical Analysis and Applications 454* (2) (2017), 1114–1128.

[4] Al-Gharabli, M. M. and Messaoudi, S. A., Existence and a general decay result for a plate equation with nonlinear damping and a logarithmic source term, *Journal of Evolution Equations 18* (1) (2018), 105–125.

[5] Bartkowski, K. and Górka, P., One-dimensional klein–gordon equation with logarithmic nonlinearities, *Journal of Physics A: Mathematical and Theoretical 41* (35) (2008), 355201.

[6] Bialynicki-Birula, I. and Mycielski, J., Wave equations with logarithmic nonlinearities, *Bull. Acad. Polon. Sci. Cl 3* (23) (1975), 461.

[7] Cazenave, T. and Haraux, A., Équations d'évolution avec non linéarité logarithmique, In *Annales de la Faculté des sciences de Toulouse: Mathématiques* (1980), vol. 2, pp. 21–51.

[8] De Martino, S., Falanga, M., Godano, C. and Lauro, G., Logarithmic schrödinger-like equation as a model for magma transport, *EPL (Europhysics Letters) 63* (3) (2003), 472.

[9] Górka, P., Logarithmic klein-gordon equation., *Acta Physica Polonica B 40* (1) (2009).

[10] Gross, L., Logarithmic sobolev inequalities, *American Journal of Mathematics 97* (4) (1975), 1061–1083.

[11] Han, X., Global existence of weak solutions for a logarithmic wave equation arising from q-ball dynamics, *Bull. Korean Math. Soc 50* (1) (2013), 275–283.

[12] Martinez, P., A new method to obtain decay rate estimates for dissipative systems, *ESAIM: Control, Optimisation and Calculus of Variations 4* (1999), 419–444.

[13] Peyravi, A., General stability and exponential growth for a class of semi-linear wave equations with logarithmic source and memory terms, *Applied Mathematics & Optimization 81* (2) (2020), 545–561.

[14] Pişkin, E., *Sobolev spaces*, Seçkin Publishing, 2017.

[15] Pişkin, E. and Irkıl, N., Mathematical behavior of solutions of fourth-order hyperbolic equation with logarithmic source term, In *Conference Proceedings of Science and Technology* (2019), vol. 2, pp. 27–36.

[16] Pişkin, E. and Irkıl, N., Well-posedness results for a sixth-order logarithmic boussinesq equation, *Filomat 33* (13) (2019), 3985–4000.