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Nonexistence of Global Solutions for the Strongly Damped Wave Equation with Variable Coefficients

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Abstract

In this work, we deal with the wave equation with variable coefficients. Under proper conditions on variable coefficients, we prove the nonexistence of global solutions.

1. Introduction

In this paper, we are concerned with the following problem:

$$\begin{cases} u_{tt} - \Delta u - \Delta u_t + \mu_1(t)|u_t|^{p-2} u_t = \mu_2(t)|u|^{q-2} u, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^n $(n \in N)$, with a smooth boundary $\partial\Omega$, $p \geq 2$, $q > 2$, $\mu_1(t)$ is a non-negative function of t and $\mu_2(t)$ is a positive functions of t. The quantity $|u_t|^{p-2} u_t$ is a damping term which assures global existence, and $|u|^{q-2} u$ is the source term which contributes to nonxistence of global solutions. $\mu_1(t)$ and $\mu_2(t)$ can be regarded as two control buttons which can dominate the polarity between damping term and source term.
In the absence of the strong damping term Δu_t, and $\mu_1(t) = \mu_2(t) \equiv 1$, then the problem (1.1) can be reduced to the following wave equation

$$u_{tt} - \Delta u + |u_t|^{p-2} u_t = |u|^{q-2} u.$$

Many authors established the existence, nonexistence and decay of solutions, see [1–6]. The interaction between nonlinear damping $(|u_t|^{p-2} u_t)$ and the source term $(|u|^{q-2} u)$ makes the problem more interesting. Levine [2,3] first studied the interaction between the linear damping $(p = 2)$ and source term by using Concavity method. But this method can't be applied in the case of a nonlinear damping term. Georgiev and Todorova [1] extended Levine's result to the nonlinear case $(p > 2)$. They showed that solutions with negative initial energy blow up in finite time. Later, Vitillaro in [6] extended these results to situations where the nonlinear damping and the solution has positive initial energy.
In [7], Yu investigated the equation with constant coefficients

$$u_{tt} - \Delta u - \Delta u_t + |u_t|^{p-2} u_t = |u|^{q-2} u. \tag{1.2}$$

He showed globality, boundedness, blow-up, convergence up to a subsequence towards the equilibria and exponential stability. Gerbi and Said-Houari [8] proved exponential decay of solutions (1.2) for $p = 2$.

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Zheng et al. [9] considered the Petrovsky equation

$$u_{tt}+\Delta^2 u+k_1(t)|u_t|^{m-2}u_t=k_2(t)|u|^{p-2}u$$

in a bounded domain. They proved the blow up of solutions.
In this paper, we established the nonexistence of solutions. To our best knowledge, the nonexistence of solutions of the wave equation with variable coefficients not yet studied.
This paper is organized as follows: In the next section, we present some lemmas, notations and local existence theorem. In section 3, the nonexistence of global solutions are given.

2. Preliminaries

In order to state the main results to problem (1.1) more clearly, we start to our work by introducing some notations and lemmas which will be used in this paper. Throughout this paper $\|u\|_p=\|u\|_{L^p(\Omega)}$ and $\|u\|_2=\|u\|$ denote the usual $L^p(\Omega)$ norm and $L^2(\Omega)$ norm, respectively. Also, $W_0^{m,2}(\Omega)=H_0^m(\Omega)$ is a Hilbert spaces (see [10, 11], for details).

Lemma 2.1. *[4]. Assume that*

$$\begin{cases} 2\le q<\infty, & n\le 2,\\ 2<q<\frac{2(n-1)}{n-2}, & n\ge 3.\end{cases}$$

Then, there exist a positive constant $C>1$, depending on Ω only, such that

$$\|u\|_q^s\le C\left(\|\nabla u\|^2+\|u\|_q^q\right) \tag{2.1}$$

for any $u\in H_0^1(\Omega)$ and $2\le s\le q$.

Lemma 2.2. *Assume that $p\ge 2$, $q>2$, $\mu_1(t)$ is a nonnegative function of t, $\mu_2(t)$ is a positive functions of t and $\mu_2'(t)\ge 0$. Let $u(t)$ be a solution of problem (1.1) then the energy functional $E(t)$ is non-increasing, namely $E'(t)\le 0$.*

Proof. Multiplying the equation (1.1) with u_t and integrating with respect to x over the domain Ω, we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\nabla u\|^2-\frac{\mu_2(t)}{q}\|u\|_q^q\right)=-\mu_1(t)\|u_t\|_p^p-\|\nabla u_t\|^2-\frac{\mu_2'(t)}{q}\|u\|_q^q. \tag{2.2}$$

By the equality (2.2), we get

$$E'(t)=-\mu_1(t)\|u_t\|_p^p-\|\nabla u_t\|^2-\frac{\mu_2'(t)}{q}\|u\|_q^q\le 0,$$

and $E(t)\le E(0)$, where

$$E(t)=\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\nabla u\|^2-\frac{\mu_2(t)}{q}\|u\|_q^q, \tag{2.3}$$

and

$$E(0)=\frac{1}{2}\|u_1\|^2+\frac{1}{2}\|\nabla u_0\|^2-\frac{\mu_2(0)}{q}\|u_0\|_q^q.$$

□

In order to obtain our main results, we set

$$H(t)=-E(t). \tag{2.4}$$

In the following remark, C denotes a generic constant that varies from line to line. Combining (2.1), (2.3) and (2.4), we obtain

Remark 2.3. *Assume that*

$$\begin{cases} 2\le q<\infty, & n\le 2,\\ 2<q<\frac{2(n-1)}{n-2}, & n\ge 3\end{cases}$$

and energy functional $E(t)<0$. Then, there exist a positive constant C, depending only on Ω, such that

$$\|u\|_q^s\le C\left(H(t)+\|u_t\|^2+\left(\frac{\mu_2(t)}{q}+1\right)\|u\|_q^q\right) \tag{2.5}$$

for any $u\in H_0^1(\Omega)$ and $2\le s\le q$.

Next, we state the local existence theorem that can be established by combining arguments of [1, 12].

Theorem 2.4. *(Local existence). Suppose that*

$$\begin{cases} 2\le q<\infty, & n\le 2,\\ 2<q<\frac{2(n-1)}{n-2}, & n\ge 3.\end{cases}$$

Then, for any given $(u_0,u_1)\in\left(H_0^1(\Omega)\times L^2(\Omega)\right)$, the problem (1.1) has a local solution satisfying

$$u\in C\left([0,T]:H_0^1(\Omega),u_t\in C\left([0,T];L^2(\Omega)\right)\cap L^p(\Omega,[0,T])\right)$$

for some $T>0$.

3. Nonexistence of Global Solutions

In this section, we will consider the nonexistence of global solutions for the problem (1.1). By using the same techniques as in [9].

Theorem 3.1. *Let the assumptions of Lemma 2.2 hold. And assume that $\mu_1(t)$ is a nonnegative function of t, $\mu_2(t)$ is a positive functions of t, $\mu_2'(t) \geq 0$ and*

$$\lim_{t\to\infty} \mu_1(t)\mu_2(t)^{\alpha(p-1)}$$

exists, where

$$0 < \alpha \leq \min\left\{\frac{q-2}{2q}, \frac{q-p}{q(p-1)}\right\}.$$

Then the solution of Eq. (1.1) blows up in finite time T^* and

$$T^* \leq \frac{1-\alpha}{\alpha\gamma L^{\frac{\alpha}{1-\alpha}}(0)}$$

if $q > p$ and the initial energy function

$$E(0) < 0,$$

where

$$L(0) = [H(0)]^{1-\alpha} + \varepsilon\int_\Omega u_0 u_1 dx > 0.$$

Proof. From (2.2)-(2.4), we have

$$\frac{d}{dt}H(t) = \mu_1(t)\|u_t\|_p^p + \|\nabla u_t\|^2 + \frac{\mu_2'(t)}{q}\|u\|_q^q \geq 0 \tag{3.1}$$

for almost, every $t \in [0,T)$. Therefore

$$0 < H(0) \leq H(t) \leq \frac{\mu_2(t)}{q}\|u\|_q^q,\ t \in [0,T). \tag{3.2}$$

Define

$$L(t) = H^{1-\alpha}(t) + \varepsilon\int_\Omega u u_t dx + \frac{\varepsilon}{2}\|\nabla u\|^2 \tag{3.3}$$

where $\varepsilon > 0$ is small to be chosen later, and

$$0 < \alpha \leq \min\left\{\frac{q-2}{2q}, \frac{q-p}{q(p-1)}\right\}. \tag{3.4}$$

Differentiating (3.3) with respect to t and combining the first equation of (1.1), we have

$$\begin{aligned} L'(t) &= (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\int_\Omega\left(uu_{tt} + u_t^2\right)dx + \varepsilon\int \nabla u\nabla u_t dx \\ &= (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\int \nabla u\nabla u_t dx \\ &\quad + \varepsilon\int_\Omega\left(u\Delta u + u\Delta u_t - \mu_1(t)|u_t|^{p-1}u + \mu_2(t)u^q + u_t^2\right)dx \\ &= (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\|u_t\|^2 - \varepsilon\|\nabla u\|^2 \\ &\quad + \varepsilon\mu_2(t)\|u\|_q^q - \varepsilon\mu_1(t)\int_\Omega |u_t|^{p-1}u dx. \end{aligned} \tag{3.5}$$

Due to the Hölder's and Young's inequalities, we have

$$\begin{aligned} \left|\mu_1(t)\int_\Omega |u_t|^{p-1}u dx\right| &\leq \mu_1(t)\int_\Omega |u_t|^{p-1}u dx \\ &\leq \left(\int_\Omega \mu_1(t)|u_t|^p dx\right)^{\frac{p-1}{p}}\left(\int_\Omega \mu_1(t)|u|^p dx\right)^{\frac{1}{p}} \\ &\leq \frac{p-1}{p}\mu_1(t)\delta^{-\frac{p}{p-1}}\|u_t\|_p^p + \frac{\delta^p}{p}\mu_1(t)\|u\|_p^p, \end{aligned} \tag{3.6}$$

where δ is positive constant to be determined later. According to the conditions $\mu_1(t) \geq 0, \mu_2'(t) \geq 0$ and (3.1), we get

$$H'(t) \geq \mu_1(t)\|u_t\|_p^p. \tag{3.7}$$

Combining (2.3), (2.4), (3.5), (3.6) and (3.7), we have

$$\begin{aligned} L'(t) \geq & \left[(1-\alpha)H^{-\alpha}(t) - \frac{p-1}{p}\varepsilon\delta^{-\frac{p}{p-1}}\right]H'(t) \\ & +\varepsilon\left(qH(t) - \frac{\delta^p}{p}\mu_1(t)\|u_t\|_p^p\right) \\ & +\varepsilon\left(\frac{q}{2}+1\right)\|u_t\|^2 + \varepsilon\left(\frac{q}{2}-1\right)\|\nabla u\|^2. \end{aligned} \tag{3.8}$$

Since the integral is taken over the variable x, it is reasonable to take δ depending on variable t. From (3.2), we obtain

$$0 < H^{-\alpha}(t) \leq H^{-\alpha}(0),$$

for every $t > 0$. Hence $H^{-\alpha}(t)$ is a positive function and bounded. Thus, by taking $\delta^{-\frac{p}{p-1}} = mH^{-\alpha}(t)$, for large m to be specified later, and substituting in (3.8), we get

$$\begin{aligned} L'(t) \geq & \left[(1-\alpha) - \frac{p-1}{p}\varepsilon m\right]H^{-\alpha}(t)H'(t) \\ & +\varepsilon\left(\frac{q}{2}+1\right)\|u_t\|^2 + \varepsilon\left(\frac{q}{2}-1\right)\|\nabla u\|^2 \\ & +\varepsilon\left[qH(t) - \frac{m^{1-p}}{p}\mu_1(t)H^{\alpha(p-1)}(t)\|u\|_p^p\right]. \end{aligned} \tag{3.9}$$

By using the (2.3), (2.4), (3.2) and the embedding $L^q(\Omega) \hookrightarrow L^p(\Omega)\ (q > p)$, we arrive at $\|u\|_p^p \leq C\|u\|_q^p$ and

$$\begin{aligned} L'(t) \geq & \left[(1-\alpha) - \frac{p-1}{p}\varepsilon m\right]H^{-\alpha}(t)H'(t) \\ & +\varepsilon\left(\frac{q}{2}+1\right)\|u_t\|^2 + \varepsilon\left(\frac{q}{2}-1\right)\|\nabla u\|^2 \\ & +\varepsilon\left[qH(t) - \frac{Cm^{1-p}}{p}\mu_1(t)\left(\frac{\mu_2(t)}{q}\right)^{\alpha(p-1)}\|u\|_q^{p+q\alpha(p-1)}\right]. \end{aligned} \tag{3.10}$$

From (3.4), we get $2 \leq s = p + q\alpha(p-1) \leq q$. Combining (2.3), (2.4), Remark 2.3 and (3.10), we obtain

$$\begin{aligned} L'(t) \geq & \left[(1-\alpha) - \frac{p-1}{p}\varepsilon m\right]H^{-\alpha}(t)H'(t) + \varepsilon\left(\frac{q}{2}+1\right)\|u_t\|^2 + \varepsilon\left(\frac{q}{2}-1\right)\|\nabla u\|^2 \\ & +\varepsilon\left[qH(t) - C_1m^{1-p}\mu_2(t)^{\alpha(p-1)}\mu_1(t)\left(H(t) + \|u_t\|_2^2 + \frac{\mu_2(t)}{q} + 1\right)\|u\|_q^q\right] \\ \geq & \left[(1-\alpha) - \frac{p-1}{p}\varepsilon m\right]H^{-\alpha}(t)H'(t) + \varepsilon\left(\frac{q+2}{2} - C_1m^{1-p}\mu_2(t)^{\alpha(p-1)}\mu_1(t)\right)H(t) \\ & +\varepsilon\left[\frac{q+6}{4} - C_1m^{1-p}\mu_2(t)^{\alpha(p-1)}\mu_1(t)\right]\|u_t\|^2 \\ & +\varepsilon\left[\frac{q-2}{2q}\mu_2(t) - C_1m^{1-p}\mu_2(t)^{\alpha(p-1)}\mu_1(t)\left(\frac{\mu_2(t)}{q} + 1\right)\right]\|u\|_q^q, \end{aligned} \tag{3.11}$$

where $C_1 = \frac{C}{pq^{\alpha(p-1)}}$. Since $\lim_{t\to\infty}\mu_1(t)\mu_2(t)^{\alpha(p-1)}$ exists, $\mu_1(t)\mu_2(t)^{\alpha(p-1)}$ is bounded for every $t > 0$. Then, we choose m large enough so that the coefficients of $H(t)$, $\|u_t\|^2$ and $\|u\|_q^q$ in (3.11) are strictly positive. Therefore, we arrive at

$$\begin{aligned} L'(t) \geq & \left[(1-\alpha) - \frac{p-1}{p}\varepsilon m\right]H^{-\alpha}(t)H'(t) \\ & +\varepsilon\beta\left[H(t) + \|u_t\|_2^2 + \left(\frac{\mu_2(t)}{q} + 1\right)\|u\|_q^q\right], \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} \beta = \min\Bigg\{ & \frac{q+2}{2} - C_1m^{1-p}\mu_2(t)^{\alpha(p-1)}\mu_1(t), \\ & \frac{q+6}{4} - C_1m^{1-p}\mu_2(t)^{\alpha(p-1)}\mu_1(t), \\ & \frac{q-2}{2q}\mu_2(t) - C_1m^{1-p}\mu_2(t)^{\alpha(p-1)}\mu_1(t)\Bigg\} \end{aligned}$$

is the minimum of the coefficients of $H(t)$, $\|u_t\|^2$ and $\|u\|_q^q$. Once m is fixed, we can take ε small enough so that $1 - \alpha - \frac{p-1}{p}\varepsilon m \geq 0$ and

$$L(0) = H^{1-\alpha}(0) + \varepsilon\int_\Omega u_0u_1dx > 0. \tag{3.13}$$

Then (3.12) becomes

$$L'(t) \geq \varepsilon\beta\left[H(t)+\|u_t\|_2^2+\left(\frac{\mu_2(t)}{q}+1\right)\|u\|_q^q\right] \geq 0. \tag{3.14}$$

Then, we have

$$L(t) \geq L(0) > 0. \tag{3.15}$$

For the definition of $L(t)$ (see (3.3)) we have

$$\begin{aligned}\left|\int_\Omega uu_t dx\right| &\leq \|u\|\,\|u_t\| \\ &\leq C\|u\|_q\,\|u_t\|\end{aligned} \tag{3.16}$$

using Hölder's inequality and the embedding $L^q(\Omega) \hookrightarrow L^p(\Omega)$ $(q>p)$. Thanks to Young's inequality, we have

$$\begin{aligned}\left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\alpha}} &\leq C\|u\|_q^{\frac{1}{1-\alpha}}\|u_t\|^{\frac{1}{1-\alpha}} \\ &\leq C\left(\|u\|_q^{\frac{2}{1-2\alpha}}+\|u_t\|^2\right)\end{aligned} \tag{3.17}$$

from (3.4), we arrive at $\frac{2}{1-2\alpha}<q$.
Combining (3.17) and Remark 2.3, we get

$$\left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\alpha}} \leq C\left(H(t)+\|u_t\|_2^2+\left(\frac{\mu_2(t)}{q}+1\right)\|u\|_q^q\right). \tag{3.18}$$

Therefore, we obtain

$$\begin{aligned}L^{\frac{1}{1-\alpha}}(t) &= \left[H^{1-\alpha}(t)+\varepsilon\int_\Omega uu_t dx\right]^{\frac{1}{1-\alpha}} \\ &\leq 2^{\frac{1}{1-\alpha}}\left(H(t)+\left|\varepsilon\int_\Omega uu_t dx\right|^{\frac{1}{1-\alpha}}\right) \\ &\leq C\left(H(t)+\|u_t\|_2^2+\left(\frac{\mu_2(t)}{q}+1\right)\|u\|_q^q\right).\end{aligned} \tag{3.19}$$

Combining (3.14), (3.15) and (3.19), we have

$$L'(t) \geq \gamma L^{\frac{1}{1-\alpha}}(t) \tag{3.20}$$

where γ is a constant depending only on C, β and ε. Integrating (3.20), we arrive at

$$L^{\frac{1}{1-\alpha}}(t) \geq \frac{1}{L^{-\frac{\alpha}{1-\alpha}}(0)-\frac{\alpha}{1-\alpha}\gamma t}. \tag{3.21}$$

If

$$t \to \left[\frac{1-\alpha}{\alpha\gamma L^{\frac{\alpha}{1-\alpha}}(0)}\right]^-, \quad L^{-\frac{\alpha}{1-\alpha}}(0)-\frac{\alpha}{1-\alpha}\gamma t \to 0.$$

Hence, $L(t)$ blows up in finite time T^* and

$$T^* \leq \frac{1-\alpha}{\alpha\gamma L^{\frac{\alpha}{1-\alpha}}(0)},$$

which complete the proof of the Theorem. □

4. Conclusion

In this paper, we obtained the nonexistence of global solutions for a strongly damped wave equation with variable coefficients. This improves and extends many results in the literature.

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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