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Article

Applications Laguerre Polynomials for Families of Bi-Univalent Functions Defined with (p, q)-Wanas Operator

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Abstract: In current manuscript, using Laguerre polynomials and $(p-q)$-Wanas operator, we identify upper bounds $|a_2|$ and $|a_3|$ which are first two Taylor-Maclaurin coefficients for a specific bi-univalent functions classes $\mathcal{W}_\Sigma(\eta, \delta, \lambda, \sigma, \theta, \alpha, \beta, p, q; h)$ and $\mathcal{K}_\Sigma(\xi, \rho, \sigma, \theta, \alpha, \beta, p, q; h)$ which cover the convex and starlike functions. Also, we discuss Fekete-Szegö type inequality for defined class.

Keywords: bi-univalent function; Fekete-Szegö problem; coefficient bound; Laguerre polynomial; (p, q)-Wanas operator; subordination

MSC: 30C45; 30C80

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1. Introduction

Denote by $\mathcal{A}$ function collections that have the style:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}, \tag{1}$$

holomorphic in $\mathbb{D} = \{z : |z| < 1\}$ in the complex plane $\mathbb{C}$.

Further, present by $\mathcal{S}$ the sub-set of $\mathcal{A}$ including of univalent functions in $\mathbb{D}$ fullfiling (1). Taking account the Koebe $\frac{1}{4}$ theorem (see [1]), each $f \in \mathcal{S}$ has an inverse f^{-1} with the properties $f^{-1}(f(z)) = z$, for $z \in \mathbb{D}$ and $f(f^{-1}(w)) = w$, with $|w| < r_0(f)$, where $r_0(f) \geq \frac{1}{4}$. If f is of the style (1), then

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right)w^4 + \cdots, \quad |w| < r_0(f). \tag{2}$$

When f and f^{-1} are univalent functions, $f \in \mathcal{A}$ is bi-univalent in $\mathbb{D}$. The set of bi-univalent functions can be expressed by Σ. The work on bi-univalent functions have been brightened by Srivastava et al. [2] in recent years. The following functions can be examplified for functions in the set of bi-univalent.

$$\frac{z}{1-z}, \quad -\log(1-z) \quad \text{and} \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

Although Koebe function is not an element of bi-univalent set of functions, the Σ is not null set.

Later, such studies continued by Ali et al. [3], Bulut et al. [4], Srivastava et al. [5] and others (see, for example, [6–18]). However, non decisive predictions of the $|a_2|$ and $|a_3|$

coefficients in given by (1) were declared in different studies. Generalized inequalities on Taylor-Maclaurin coefficients

$$|a_n| \qquad (n \in \mathbb{N};\ n \geqq 3)$$

for $f \in \Sigma$ has not been totally solved yet for several subfamilies of the Σ.

$|a_3 - \mu a_2^2|$ of the Fekete-Szegö function for $f \in \mathcal{S}$ is well-known in the Geometric Function Theory.

Its origin lies in the refutation of the Littlewood-Paley conjecture by Fekete-Szegö [19]. In that case, the coefficients of odd (single-valued) univalent functions are bounded by unity.

Functions have received much attention since then, especially in the investigation of many subclasses of the single-valued function family.

This topic has become very interesting for Geometric Function Theorists (see for example [20–25]).

The generator function for Laguerre polynomial $L_n^{\gamma}(\tau)$ is the polynomial answer $\phi(\tau)$ of the differential equation ([26])

$$\tau\phi'' + (1 + \gamma - \tau)\phi' + n\phi = 0,$$

where $\gamma > -1$ and n is non-negative integers.

The generating function of generator function for Laguerre polynomial $L_n^{\gamma}(\tau)$ is expressed as below:

$$H_{\gamma}(\tau, z) = \sum_{n=0}^{\infty} L_n^{\gamma}(\tau) z^n = \frac{e^{-\frac{\tau z}{1-z}}}{(1-z)^{\gamma+1}}, \tag{3}$$

where $\tau \in \mathbb{R}$ and $z \in \mathbb{D}$. The generator function for Laguerre polynomial can also be expressed given below:

$$L_{n+1}^{\gamma}(\tau) = \frac{2n + 1 + \gamma - \tau}{n+1} L_n^{\gamma}(\tau) - \frac{n+\gamma}{n+1} L_{n-1}^{\gamma}(\tau) \quad (n \geq 1),$$

with the initial terms

$$L_0^{\gamma}(\tau) = 1, \quad L_1^{\gamma}(\tau) = 1 + \gamma - \tau \quad \text{and} \quad L_2^{\gamma}(\tau) = \frac{\tau^2}{2} - (\gamma+2)\tau + \frac{(\gamma+1)(\gamma+2)}{2}. \tag{4}$$

Simply, when $\gamma = 0$ the generator function for Laguerre polynomial leads to the simply Laguerre polynomial, $L_n^0(\tau) = L_n(\tau)$.

Let f and g be holomorphic in $\mathbb{D}$, it is clear that f is subordinate to g, if there occurs a holomorphic function w in $\mathbb{D}$ such that $w(0) = 0$, and $|w(z)| < 1$, for $z \in \mathbb{D}$ so that $f(z) = g(w(z))$. This subordination is indicated by $f \prec g$. Moreover, if g is univalent in $\mathbb{D}$, then we have the balance (see [27]), given by $f(z) \prec g(z) \iff f(\mathbb{D}) \subset g(\mathbb{D})$ *and* $f(0) = g(0)$.

The (p,q)-derivative operator or (p,q)-difference operator $(0 < q < p \leq 1)$, for a function f is stated by

$$D_{p,q}f(z) = \frac{f(pz) - f(qz)}{(p-q)z} \qquad (z \in \mathbb{D}^* = \mathbb{D} \setminus \{0\}),$$

and

$$D_{p,q}f(0) = f'(0).$$

More information on the subject of (p,q)-calculus are founded in [28–33].

For $f \in \mathcal{A}$, we conclude that

$$D_{p,q}f(z) = 1 + \sum_{n=2}^{\infty} [n]_{p,q} a_n z^{n-1},$$

where the (p,q)-bracket number or twin-basic $[n]_{p,q}$ is showed by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \cdots + pq^{n-2} + q^{n-1} \quad (p \neq q),$$

which is a native generator number for q, namely is, we get (see [34,35])

$$\lim_{p \to 1^-} [n]_{p,q} = [n]_q = \frac{1 - q^n}{1 - q}.$$

Obviously, the impression $[n]_{p,q}$ is symmetric, namely,

$$[n]_{p,q} = [n]_{q,p}.$$

Wanas and Cotîrlă [36] presented $W^{\sigma,\theta}_{\alpha,\beta,p,q} : \mathcal{A} \longrightarrow \mathcal{A}$ known as $(p-q)$-Wanas operator showed by

$$W^{\sigma,\theta}_{\alpha,\beta,p,q} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{[\Psi_n(\sigma,\alpha,\beta)]_{p,q}}{[\Psi_1(\sigma,\alpha,\beta)]_{p,q}} \right)^{\theta} a_n z^n = z + \sum_{n=2}^{\infty} \frac{[\Psi_n(\sigma,\alpha,\beta)]^{\theta}_{p,q}}{[\Psi_1(\sigma,\alpha,\beta)]^{\theta}_{p,q}} a_n z^n,$$

where

$$\Psi_n(\sigma,\alpha,\beta) = \sum_{\tau=1}^{\sigma} \binom{\sigma}{\tau} (-1)^{\tau+1} (\alpha^{\tau} + n\beta^{\tau}),\ \Psi_1(\sigma,\alpha,\beta) = \sum_{\tau=1}^{\sigma} \binom{\sigma}{\tau} (-1)^{\tau+1} (\alpha^{\tau} + \beta^{\tau}),$$

and

$$\alpha \in \mathbb{R}, \beta \in \mathbb{R}_0^+ \text{ with } \alpha + \beta > 0, n - 1 \in \mathbb{N}, \sigma \in \mathbb{N}, \theta \in \mathbb{N}_0, 0 < q < p \leq 1 \text{ and } z \in \mathbb{D}.$$

Remark 1. *The operator $W^{\sigma,\theta}_{\alpha,\beta,p,q}$ is a generalized form of several operators given in previous researches for some values of parameters which are mentioned below.*

1. *For $p = \sigma = \beta = 1$, $\theta = -\nu$, $\Re(\nu) > 1$ and $\alpha \in \mathbb{C} \setminus \mathbb{Z}_0^-$, the operator $W^{\sigma,\theta}_{\alpha,\beta,p,q}$ decreases to the q-Srivastava Attiya operator $J^{\nu}_{q,\alpha}$ [37].*
2. *For $p = \sigma = \beta = 1$, $\theta = -1$ and $\alpha > -1$, the operator $W^{\sigma,\theta}_{\alpha,\beta,p,q}$ decreases to the q-Bernardi operator [38].*
3. *For $p = \sigma = \alpha = \beta = 1$ and $\theta = -1$, the operator $W^{\sigma,\theta}_{\alpha,\beta,p,q}$ decreases to the q-Libera operator [38].*
4. *For $\alpha = 0$ and $p = \sigma = \beta = 1$, the operator $W^{\sigma,\theta}_{\alpha,\beta,p,q}$ decreases to the q-Sălăgean operator [39].*
5. *For $q \longrightarrow 1^-$ and $p = \sigma = 1$, the operator $W^{\sigma,\theta}_{\alpha,\beta,p,q}$ decreases to the operator $I^{\theta}_{\alpha,\beta}$ was presented and studied by Swamy [40].*
6. *For $q \longrightarrow 1^-$, $p = \sigma = \beta = 1$, $\theta = -\nu$, $\Re(\nu) > 1$ and $s \in \mathbb{C} \setminus \mathbb{Z}_0^-$, the operator $W^{\sigma,\theta}_{\alpha,\beta,p,q}$ decreases to the operator J^{ν}_{α} was presented by Srivastava and Attiya [41]. The operator J^{ν}_{s} is well-known as Srivastava-Attiya operator by researchers.*
7. *For $q \longrightarrow 1^-$, $p = \sigma = \beta = 1$ and $\alpha > -1$, the operator $W^{\sigma,\theta}_{\alpha,\beta,p,q}$, decreases to the operator I^{θ}_{α} was presented by Cho and Srivastava [42].*
8. *For $q \longrightarrow 1^-$, $p = \sigma = \alpha = \beta = 1$, the operator $W^{\sigma,\theta}_{\alpha,\beta,p,q}$ decreases to the operator I^{θ} was presented by Uralegaddi and Somanatha [43].*
9. *For $q \longrightarrow 1^-$, $p = \sigma = \alpha = \beta = 1$, $\theta = -\xi$ and $\xi > 0$, the operator $W^{\sigma,\theta}_{\alpha,\beta,p,q}$ decreases to the operator I^{ξ} was presented by Jung et al. [44]. The operator I^{ξ} is the Jung-Kim-Srivastava integral operator.*
10. *For $q \longrightarrow 1^-$, $p = \sigma = \beta = 1$, $\theta = -1$ and $\alpha > -1$, the operator $W^{\sigma,\theta}_{\alpha,\beta,p,q}$ decreases to the Bernardi operator [45].*

11. *For $q \longrightarrow 1^-$, $\alpha = 0$, $p = \sigma = \beta = 1$ and $\theta = -1$, the operator $W^{\sigma,\theta}_{\alpha,\beta,p,q}$ decreases to the Alexander operator [46].*
12. *For $q \longrightarrow 1^-$, $p = \sigma = 1$, $\alpha = 1 - \beta$ and $t \geq 0$, the operator $W^{\sigma,\theta}_{\alpha,\beta,p,q}$ decreases to the operator D^{θ}_{β} was presented by Al-Oboudi [19].*
13. *For $q \longrightarrow 1^-$, $p = \sigma = 1$, $\alpha = 0$ and $\beta = 1$, the operator $W^{\sigma,\theta}_{\alpha,\beta,p,q}$ decreases to the operator S^{θ} was presented by Sălăgean [47].*

2. Main Results

Firstly, We start to present the classes $\mathcal{W}_\Sigma(\eta,\delta,\lambda,\sigma,\theta,\alpha,\beta,p,q;h)$ and $\mathcal{K}_\Sigma(\xi,\rho,\sigma,\theta,\alpha,\beta,p,q;h)$ given below:

Definition 1. *Suppose that $0 \leq \eta \leq 1$, $0 \leq \lambda \leq 1$, $0 \leq \delta \leq 1$ and h is analytic in $\mathbb{D}$, $h(0) = 1$. $f \in \Sigma$ is in the class $\mathcal{W}_\Sigma(\eta,\delta,\lambda,\sigma,\theta,\alpha,\beta,p,q;h)$ if it provides the subordinations:*

$$\left(\frac{z\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f(z)\right)'}{W^{\sigma,\theta}_{\alpha,\beta,p,q}f(z)}\right)^{\eta}\left[(1-\delta)\frac{z\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f(z)\right)'}{W^{\sigma,\theta}_{\alpha,\beta,p,q}f(z)}+\delta\left(1+\frac{z\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f(z)\right)''}{\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f(z)\right)'}\right)\right]^{\lambda}\prec h(z)$$

and

$$\left(\frac{w\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f^{-1}(w)\right)'}{W^{\sigma,\theta}_{\alpha,\beta,p,q}f^{-1}(w)}\right)^{\eta}\left[(1-\delta)\frac{w\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f^{-1}(w)\right)'}{W^{\sigma,\theta}_{\alpha,\beta,p,q}f^{-1}(w)}+\delta\left(1+\frac{w\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f^{-1}(w)\right)''}{\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f^{-1}(w)\right)'}\right)\right]^{\lambda}\prec h(w),$$

where f^{-1} is given by (2).

Definition 2. *Suppose that $0 \leq \xi \leq 1$, $0 \leq \rho < 1$ and h is analytic in $\mathbb{D}$, $h(0) = 1$. $f \in \Sigma$ is in the class $\mathcal{K}_\Sigma(\xi,\rho,\sigma,\theta,\alpha,\beta,p,q;h)$ if it provides the subordinations:*

$$(1-\xi)\frac{z\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f(z)\right)'}{(1-\rho)W^{\sigma,\theta}_{\alpha,\beta,p,q}f(z)+\rho z\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f(z)\right)'}$$
$$+\xi\left(\frac{\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f(z)\right)'+z\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f(z)\right)''}{\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f(z)\right)'+\rho z\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f(z)\right)''}\right)\prec h(z)$$

and

$$(1-\xi)\frac{w\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f^{-1}(w)\right)'}{(1-\rho)W^{\sigma,\theta}_{\alpha,\beta,p,q}f^{-1}(w)+\rho w\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f^{-1}(w)\right)'}$$
$$+\xi\left(\frac{\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f^{-1}(w)\right)'+w\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f^{-1}(w)\right)''}{\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f^{-1}(w)\right)'+\rho w\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f^{-1}(w)\right)''}\right)\prec h(w),$$

where f^{-1} is given by (2).

Theorem 1. *Suppose that $0 \leq \eta \leq 1$, $0 \leq \lambda \leq 1$ and $0 \leq \delta \leq 1$. If $f \in \Sigma$ of the style (1) be an element of class $\mathcal{W}_\Sigma(\eta,\delta,\lambda,\sigma,\theta,\alpha,\beta,p,q;h)$, with $h(z) = 1 + e_1 z + e_2 z^2 + \cdots$, then*

$$|a_2| \leq \frac{(\eta+\lambda(\delta+1))[\Psi_2(\sigma,\alpha,\beta)]^{\theta}_{p,q}|e_1|}{[\Psi_1(\sigma,\alpha,\beta)]^{\theta}_{p,q}} = \frac{|e_1|}{\Omega}$$

and

$$|a_3| \leq \min\left\{\max\left\{\left|\frac{e_1}{\Delta}\right|, \left|\frac{e_2}{\Delta} - \frac{\varphi e_1^2}{\Omega^2\Delta}\right|\right\}, \max\left\{\left|\frac{e_1}{\Delta}\right|, \left|\frac{e_2}{\Delta} - \frac{(2\Delta+\varphi)e_1^2}{\Omega^2\Delta}\right|\right\}\right\}, \tag{5}$$

where

$$\begin{aligned}
\Omega &= \frac{(\eta+\lambda(\delta+1))[\Psi_2(\sigma,\alpha,\beta)]_{p,q}^{\theta}}{[\Psi_1(\sigma,\alpha,\beta)]_{p,q}^{\theta}}, \\
\Delta &= \frac{2(\eta+\lambda(2\delta+1))[\Psi_3(\sigma,\alpha,\beta)]_{p,q}^{\theta}}{[\Psi_1(\sigma,\alpha,\beta)]_{p,q}^{\theta}}, \\
\varphi &= \frac{[\eta(\eta-1)+\lambda(\delta+1)(2\eta+(\lambda-1)(\delta+1))-2(\eta+\lambda(3\delta+1))][\Psi_2(\sigma,\alpha,\beta)]_{p,q}^{2\theta}}{2[\Psi_1(\sigma,\alpha,\beta)]_{p,q}^{2\theta}}.
\end{aligned} \tag{6}$$

Proof. Assume that $f \in \mathcal{W}_{\Sigma}(\eta,\delta,\lambda,\sigma,\theta,\alpha,\beta,p,q;e_1;e_2)$. Then there consists two holomorphic functions $\phi,\psi : \mathbb{D} \longrightarrow \mathbb{D}$ showed by

$$\phi(z) = r_1 z + r_2 z^2 + r_3 z^3 + \cdots \quad (z \in \mathbb{D}) \tag{7}$$

and

$$\psi(w) = s_1 w + s_2 w^2 + s_3 w^3 + \cdots \quad (w \in \mathbb{D}), \tag{8}$$

with $\phi(0) = \psi(0) = 0$, $|\phi(z)| < 1$, $|\psi(w)| < 1$, $z,w \in \mathbb{D}$ so that

$$\begin{aligned}
&\left(\frac{z\left(W_{\alpha,\beta,p,q}^{\sigma,\theta}f(z)\right)'}{W_{\alpha,\beta,p,q}^{\sigma,\theta}f(z)}\right)^{\eta}\left[(1-\delta)\frac{z\left(W_{\alpha,\beta,p,q}^{\sigma,\theta}f(z)\right)'}{W_{\alpha,\beta,p,q}^{\sigma,\theta}f(z)} + \delta\left(1 + \frac{z\left(W_{\alpha,\beta,p,q}^{\sigma,\theta}f(z)\right)''}{\left(W_{\alpha,\beta,p,q}^{\sigma,\theta}f(z)\right)'}\right)\right]^{\lambda} \\
&= 1 + e_1\phi(z) + e_2\phi^2(z) + \cdots
\end{aligned} \tag{9}$$

and

$$\begin{aligned}
&\left(\frac{w\left(W_{\alpha,\beta,p,q}^{\sigma,\theta}f^{-1}(w)\right)'}{W_{\alpha,\beta,p,q}^{\sigma,\theta}f^{-1}(w)}\right)^{\eta}\left[(1-\delta)\frac{w\left(W_{\alpha,\beta,p,q}^{\sigma,\theta}f^{-1}(w)\right)'}{W_{\alpha,\beta,p,q}^{\sigma,\theta}f^{-1}(w)} + \delta\left(1 + \frac{w\left(W_{\alpha,\beta,p,q}^{\sigma,\theta}f^{-1}(w)\right)''}{\left(W_{\alpha,\beta,p,q}^{\sigma,\theta}f^{-1}(w)\right)'}\right)\right]^{\lambda} \\
&= 1 + e_1\psi(w) + e_2\psi^2(w) + \cdots.
\end{aligned} \tag{10}$$

Unification of (7), (8), (9) and (10), yield

$$\begin{aligned}
&\left(\frac{z\left(W_{\alpha,\beta,p,q}^{\sigma,\theta}f(z)\right)'}{W_{\alpha,\beta,p,q}^{\sigma,\theta}f(z)}\right)^{\eta}\left[(1-\delta)\frac{z\left(W_{\alpha,\beta,p,q}^{\sigma,\theta}f(z)\right)'}{W_{\alpha,\beta,p,q}^{\sigma,\theta}f(z)} + \delta\left(1 + \frac{z\left(W_{\alpha,\beta,p,q}^{\sigma,\theta}f(z)\right)''}{\left(W_{\alpha,\beta,p,q}^{\sigma,\theta}f(z)\right)'}\right)\right]^{\lambda} \\
&= 1 + e_1 r_1 z + \left[e_1 r_2 + e_2 r_1^2\right]z^2 + \cdots
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
&\left(\frac{w\left(W_{\alpha,\beta,p,q}^{\sigma,\theta}f^{-1}(w)\right)'}{W_{\alpha,\beta,p,q}^{\sigma,\theta}f^{-1}(w)}\right)^{\eta}\left[(1-\delta)\frac{w\left(W_{\alpha,\beta,p,q}^{\sigma,\theta}f^{-1}(w)\right)'}{W_{\alpha,\beta,p,q}^{\sigma,\theta}f^{-1}(w)} + \delta\left(1 + \frac{w\left(W_{\alpha,\beta,p,q}^{\sigma,\theta}f^{-1}(w)\right)''}{\left(W_{\alpha,\beta,p,q}^{\sigma,\theta}f^{-1}(w)\right)'}\right)\right]^{\lambda} \\
&= 1 + e_1 s_1 w + \left[e_1 s_2 + e_2 s_1^2\right]w^2 + \cdots.
\end{aligned} \tag{12}$$

It is clear that if $|\phi(z)| < 1$ and $|\psi(w)| < 1$, $z,w \in \mathbb{D}$, we obtain

$$|r_j| \leq 1 \quad \text{and} \quad |s_j| \leq 1 \; (j \in \mathbb{N}).$$

Taking into account (11) and (12), after simplifying, we find that

$$\frac{(\eta+\lambda(\delta+1))[\Psi_2(\sigma,\alpha,\beta)]^{\theta}_{p,q}}{[\Psi_1(\sigma,\alpha,\beta)]^{\theta}_{p,q}}a_2 = e_1 r_1, \tag{13}$$

$$\begin{aligned} &\frac{2(\eta+\lambda(2\delta+1))[\Psi_3(\sigma,\alpha,\beta)]^{\theta}_{p,q}}{[\Psi_1(\sigma,\alpha,\beta)]^{\theta}_{p,q}}a_3 \\ &+\frac{[\eta(\eta-1)+\lambda(\delta+1)(2\eta+(\lambda-1)(\delta+1))-2(\eta+\lambda(3\delta+1))][\Psi_2(\sigma,\alpha,\beta)]^{2\theta}_{p,q}}{2[\Psi_1(\sigma,\alpha,\beta)]^{2\theta}_{p,q}}a_2^2 \\ &= e_1 r_2 + e_2 r_1^2, \end{aligned} \tag{14}$$

$$-\frac{(\eta+\lambda(\delta+1))[\Psi_2(\sigma,\alpha,\beta)]^{\theta}_{p,q}}{[\Psi_1(\sigma,\alpha,\beta)]^{\theta}_{p,q}}a_2 = e_1 s_1 \tag{15}$$

and

$$\begin{aligned} &\frac{2(\eta+\lambda(2\delta+1))[\Psi_3(\sigma,\alpha,\beta)]^{\theta}_{p,q}}{[\Psi_1(\sigma,\alpha,\beta)]^{\theta}_{p,q}}\left(2a_2^2-a_3\right) \\ &+\frac{[\eta(\eta-1)+\lambda(\delta+1)(2\eta+(\lambda-1)(\delta+1))-2(\eta+\lambda(3\delta+1))][\Psi_2(\sigma,\alpha,\beta)]^{2\theta}_{p,q}}{2[\Psi_1(\sigma,\alpha,\beta)]^{2\theta}_{p,q}}a_2^2 \\ &= e_1 s_2 + e_2 s_1^2. \end{aligned} \tag{16}$$

If we implement notation (6), then (13) and (14) becomes

$$\Omega a_2 = e_1 r_1, \quad \Delta a_3 + \varphi a_2^2 = e_1 r_2 + e_2 r_1^2. \tag{17}$$

This gives

$$\frac{\Delta}{e_1}a_3 = r_2 + \left(\frac{e_2}{e_1} - \frac{\varphi e_1}{\Omega^2}\right) r_1^2, \tag{18}$$

and on using the given certain result ([48], p. 10):

$$|r_2 - \mu r_1^2| \le \max\{1, |\mu|\} \tag{19}$$

for every $\mu \in \mathbb{C}$, we get

$$\left|\frac{\Delta}{e_1}\right| |a_3| \le \max\left\{1, \left|\frac{e_2}{e_1} - \frac{\varphi e_1}{\Omega^2}\right|\right\}. \tag{20}$$

In the same way, (15) and (16) becomes

$$-\Omega a_2 = e_1 s_1, \quad \Delta(2a_2^2 - a_3) + \varphi a_2^2 = e_1 s_2 + e_2 s_1^2. \tag{21}$$

This gives

$$-\frac{\Delta}{e_1}a_3 = s_2 + \left(\frac{e_2}{e_1} - \frac{(2\Delta+\varphi)e_1}{\Omega^2}\right) s_1^2. \tag{22}$$

Applying (19), we obtain

$$\left|\frac{\Delta}{e_1}\right| |a_3| \le \max\left\{1, \left|\frac{e_2}{e_1} - \frac{(2\Delta+\varphi)e_1}{\Omega^2}\right|\right\}. \tag{23}$$

Inequality (5) follows from (20) and (23). □

If we take the generating function $L_n^{\gamma}(\tau)$ given by (3) common generalized Laguerre polynomials as $h(z)$, then from the equalities given(4), we get $e_1 = 1+\gamma-\tau$ and $e_2 = \frac{\tau^2}{2} - (\gamma+2)\tau + \frac{(\gamma+1)(\gamma+2)}{2}$. We obtain following corollary from Theorem 1.

Corollary 1. *If $f \in \Sigma$ given by style (1) is in the family $\mathcal{W}_\Sigma(\eta, \delta, \lambda, \sigma, \theta, \alpha, \beta, p, q; H_\gamma(\tau, z))$, then*

$$|a_2| \leq \frac{(\eta + \lambda(\delta+1))[\Psi_2(\sigma,\alpha,\beta)]^\theta_{p,q}|1+\gamma-\tau|}{[\Psi_1(\sigma,\alpha,\beta)]^\theta_{p,q}} = \frac{|1+\gamma-\tau|}{\Omega}$$

and

$$\begin{aligned} |a_3| \quad \leq \quad & \min\left\{\max\left\{\left|\frac{1+\gamma-\tau}{\Delta}\right|, \left|\frac{\frac{\tau^2}{2} - (\gamma+2)\tau + \frac{(\gamma+1)(\gamma+2)}{2}}{\Delta} - \frac{\varphi(1+\gamma-\tau)^2}{\Omega^2\Delta}\right|\right\},\right. \\ & \left.\max\left\{\left|\frac{1+\gamma-\tau}{\Delta}\right|, \left|\frac{\frac{\tau^2}{2} - (\gamma+2)\tau + \frac{(\gamma+1)(\gamma+2)}{2}}{\Delta} - \frac{(2\Delta+\varphi)(1+\gamma-\tau)^2}{\Omega^2\Delta}\right|\right\}\right\}, \end{aligned}$$

for all η, λ, δ so that $0 \leq \eta \leq 1$, $0 \leq \lambda \leq 1$ and $0 \leq \delta \leq 1$, where Ω, Δ, φ are given by (6) and $H_\gamma(\tau, z)$ is given by (3).

Theorem 2. *Suppose that $0 \leq \xi \leq 1$ and $0 \leq \rho < 1$. If $f \in \Sigma$ of the style (1) be an element of the class $\mathcal{K}_\Sigma(\xi, \rho, \sigma, \theta, \alpha, \beta, p, q; h)$, with $h(z) = 1 + e_1 z + e_2 z^2 + \cdots$, then*

$$|a_2| \leq \frac{(\xi+1)(1-\rho)[\Psi_2(\sigma,\alpha,\beta)]^\theta_{p,q}|e_1|}{[\Psi_1(\sigma,\alpha,\beta)]^\theta_{p,q}} = \frac{|e_1|}{\Upsilon}$$

and

$$|a_3| \leq \min\left\{\max\left\{\left|\frac{e_1}{\Phi}\right|, \left|\frac{e_2}{\Phi} - \frac{\chi e_1^2}{\Upsilon^2\Phi}\right|\right\}, \max\left\{\left|\frac{e_1}{\Phi}\right|, \left|\frac{e_2}{\Phi} - \frac{(2\Phi+\chi)e_1^2}{\Upsilon^2\Phi}\right|\right\}\right\}, \tag{24}$$

where

$$\begin{aligned} \Upsilon &= \tfrac{(\xi+1)(1-\rho)[\Psi_2(\sigma,\alpha,\beta)]^\theta_{p,q}}{[\Psi_1(\sigma,\alpha,\beta)]^\theta_{p,q}}, \\ \Phi &= \tfrac{2(2\xi+1)(1-\rho)[\Psi_3(\sigma,\alpha,\beta)]^\theta_{p,q}}{[\Psi_1(\sigma,\alpha,\beta)]^\theta_{p,q}}, \\ \chi &= \tfrac{(2\xi+1)(\rho^2-1)[\Psi_2(\sigma,\alpha,\beta)]^{2\theta}_{p,q}}{[\Psi_1(\sigma,\alpha,\beta)]^{2\theta}_{p,q}}. \end{aligned} \tag{25}$$

Proof. Assume that $f \in \mathcal{K}_\Sigma(\xi, \rho, \sigma, \theta, \alpha, \beta, p, q; e_1; e_2)$. Then there consists two holomorphic functions $\phi, \psi : \mathbb{D} \longrightarrow \mathbb{D}$ such that

$$\begin{aligned} & (1-\xi)\frac{z\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f(z)\right)'}{(1-\rho)W^{\sigma,\theta}_{\alpha,\beta,p,q}f(z) + \rho z\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f(z)\right)'} + \xi\left(\frac{\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f(z)\right)' + z\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f(z)\right)''}{\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f(z)\right)' + \rho z\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f(z)\right)''}\right) \\ & = 1 + e_1\phi(z) + e_2\phi^2(z) + \cdots \end{aligned} \tag{26}$$

and

$$\begin{aligned} & (1-\xi)\frac{w\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f^{-1}(w)\right)'}{(1-\rho)W^{\sigma,\theta}_{\alpha,\beta,p,q}f^{-1}(w) + \rho w\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f^{-1}(w)\right)'} + \xi\left(\frac{\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f^{-1}(w)\right)' + w\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f^{-1}(w)\right)''}{\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f^{-1}(w)\right)' + \rho w\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f^{-1}(w)\right)''}\right) \\ & = 1 + e_1\psi(w) + e_2\psi^2(w) + \cdots, \end{aligned} \tag{27}$$

where ϕ and ψ given by the style (7) and (8). Unification of (26) and (27), serve

$$(1-\xi)\frac{z\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f(z)\right)'}{(1-\rho)W^{\sigma,\theta}_{\alpha,\beta,p,q}f(z)+\rho z\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f(z)\right)'}+\xi\left(\frac{\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f(z)\right)'+z\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f(z)\right)''}{\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f(z)\right)'+\rho z\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f(z)\right)''}\right)$$
$$=1+e_1r_1z+\left[e_1r_2+e_2r_1^2\right]z^2+\cdots \tag{28}$$

and

$$(1-\xi)\frac{w\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f^{-1}(w)\right)'}{(1-\rho)W^{\sigma,\theta}_{\alpha,\beta,p,q}f^{-1}(w)+\rho w\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f^{-1}(w)\right)'}+\xi\left(\frac{\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f^{-1}(w)\right)'+w\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f^{-1}(w)\right)''}{\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f^{-1}(w)\right)'+\rho w\left(W^{\sigma,\theta}_{\alpha,\beta,p,q}f^{-1}(w)\right)''}\right)$$
$$=1+e_1s_1w+\left[e_1s_2+e_2s_1^2\right]w^2+\cdots. \tag{29}$$

It is clear that if $|\phi(z)|<1$ and $|\psi(w)|<1$, $z,w\in\mathbb{D}$, we obtain

$$|r_j|\le 1 \quad\text{and}\quad |s_j|\le 1\ (j\in\mathbb{N}).$$

Taking into account (28) and (29), after simplifying, we find that

$$\frac{(\xi+1)(1-\rho)[\Psi_2(\sigma,\alpha,\beta)]^{\theta}_{p,q}}{[\Psi_1(\sigma,\alpha,\beta)]^{\theta}_{p,q}}a_2=e_1r_1, \tag{30}$$

$$\frac{2(2\xi+1)(1-\rho)[\Psi_3(\sigma,\alpha,\beta)]^{\theta}_{p,q}}{[\Psi_1(\sigma,\alpha,\beta)]^{\theta}_{p,q}}a_3+\frac{(2\xi+1)(\rho^2-1)[\Psi_2(\sigma,\alpha,\beta)]^{2\theta}_{p,q}}{[\Psi_1(\sigma,\alpha,\beta)]^{2\theta}_{p,q}}a_2^2=e_1r_2+e_2r_1^2, \tag{31}$$

$$-\frac{(\xi+1)(1-\rho)[\Psi_2(\sigma,\alpha,\beta)]^{\theta}_{p,q}}{[\Psi_1(\sigma,\alpha,\beta)]^{\theta}_{p,q}}a_2=e_1s_1 \tag{32}$$

and

$$\frac{2(2\xi+1)(1-\rho)[\Psi_3(\sigma,\alpha,\beta)]^{\theta}_{p,q}}{[\Psi_1(\sigma,\alpha,\beta)]^{\theta}_{p,q}}\left(2a_2^2-a_3\right)+\frac{(2\xi+1)(\rho^2-1)[\Psi_2(\sigma,\alpha,\beta)]^{2\theta}_{p,q}}{[\Psi_1(\sigma,\alpha,\beta)]^{2\theta}_{p,q}}a_2^2$$
$$=e_1s_2+e_2s_1^2. \tag{33}$$

If we implement notation (25), then (30) and (31) becomes

$$\Upsilon a_2=e_1r_1,\quad \Phi a_3+\chi a_2^2=e_1r_2+e_2r_1^2. \tag{34}$$

This gives

$$\frac{\Phi}{e_1}a_3=r_2+\left(\frac{e_2}{e_1}-\frac{\chi e_1}{\Upsilon^2}\right)r_1^2, \tag{35}$$

and on using the given certain result ([48], p. 10):

$$|r_2-\mu r_1^2|\le\max\{1,|\mu|\} \tag{36}$$

for every $\mu\in\mathbb{C}$, we get

$$\left|\frac{\Phi}{e_1}\right||a_3|\le\max\left\{1,\left|\frac{e_2}{e_1}-\frac{\chi e_1}{\Upsilon^2}\right|\right\}. \tag{37}$$

In the same way, (32) and (33) becomes

$$-\Upsilon a_2=e_1s_1,\quad \Phi(2a_2^2-a_3)+\chi a_2^2=e_1s_2+e_2s_1^2. \tag{38}$$

This gives

$$-\frac{\Phi}{e_1}a_3 = s_2 + \left(\frac{e_2}{e_1} - \frac{(2\Phi+\chi)e_1}{\Upsilon^2}\right)s_1^2. \tag{39}$$

Applying (36), we obtain

$$\left|\frac{\Phi}{e_1}\right||a_3| \leq \max\left\{1, \left|\frac{e_2}{e_1} - \frac{(2\Phi+\chi)e_1}{\Upsilon^2}\right|\right\}. \tag{40}$$

Inequality (24) follows from (37) and (40). □

If we take the generating function $L_n^{\gamma}(\tau)$ given by (3) common generalized Laguerre polynomials as $h(z)$, then from the equalities given(4), we get $e_1 = 1+\gamma-\tau$ and $e_2 = \frac{\tau^2}{2} - (\gamma+2)\tau + \frac{(\gamma+1)(\gamma+2)}{2}$. We obtain following corollary from Theorem 2.

Corollary 2. *If $f \in \Sigma$ of the style (1) be an element of the class $\mathcal{K}_{\Sigma}(\xi,\rho,\sigma,\theta,\alpha,\beta,p,q;H_{\gamma}(\tau,z))$, then*

$$|a_2| \leq \frac{(\xi+1)(1-\rho)[\Psi_2(\sigma,\alpha,\beta)]_{p,q}^{\theta}|1+\gamma-\tau|}{[\Psi_1(\sigma,\alpha,\beta)]_{p,q}^{\theta}} = \frac{|1+\gamma-\tau|}{\Upsilon}$$

and

$$\begin{aligned} |a_3| \quad \leq \quad & \min\left\{\max\left\{\left|\frac{1+\gamma-\tau}{\Phi}\right|, \left|\frac{\frac{\tau^2}{2}-(\gamma+2)\tau+\frac{(\gamma+1)(\gamma+2)}{2}}{\Phi} - \frac{\chi(1+\gamma-\tau)^2}{\Upsilon^2\Phi}\right|\right\},\right. \\ & \left.\max\left\{\left|\frac{1+\gamma-\tau}{\Phi}\right|, \left|\frac{\frac{\tau^2}{2}-(\gamma+2)\tau+\frac{(\gamma+1)(\gamma+2)}{2}}{\Phi} - \frac{(2\Phi+\chi)(1+\gamma-\tau)^2}{\Upsilon^2\Phi}\right|\right\}\right\}, \end{aligned}$$

for all ξ,ρ so that $0 \leq \xi \leq 1$ and $0 \leq \rho < 1$, where Υ,Φ,χ are introduced by (25) and $H_{\gamma}(\tau,z)$ is given by (3).

We investigate the "Fekete-Szegö Inequalities" for the families $\mathcal{W}_{\Sigma}(\eta,\delta,\lambda,\sigma,\theta,\alpha,\beta,p,q;h)$ and $\mathcal{K}_{\Sigma}(\xi,\rho,\sigma,\theta,\alpha,\beta,p,q;h)$ in next theorems.

Theorem 3. *If $f \in \Sigma$ of the style (1) be an element of family $\mathcal{W}_{\Sigma}(\eta,\delta,\lambda,\sigma,\theta,\alpha,\beta,p,q;h)$, then*

$$\left|a_3 - \zeta a_2^2\right| \leq \frac{|e_1|}{\Delta}\min\left\{\max\left\{1, \left|\frac{e_2}{e_1} + \frac{(\zeta\Delta-\varphi)e_1}{\Omega^2}\right|\right\}, \max\left\{1, \left|\frac{e_2}{e_1} - \frac{(2\Delta+\varphi-\zeta\Delta)e_1}{\Omega^2}\right|\right\}\right\},$$

for all $\zeta,\eta,\lambda,\delta$ such that $\zeta \in \mathbb{R}$, $0 \leq \eta \leq 1$, $0 \leq \lambda \leq 1$ and $0 \leq \delta \leq 1$, where Ω,Δ,φ are given by (6) and e_1, e_2, a_2 and a_3 as defined in Theorem 1.

Proof. We implement the impressions from the Theorem 1's proof. From (17) and from (18), we get

$$a_3 - \zeta a_2^2 = \frac{e_1}{\Delta}\left(r_2 + \left(\frac{e_2}{e_1} + \frac{(\zeta\Delta-\varphi)e_1}{\Omega^2}\right)r_1^2\right)$$

by using the certain result $|r_2 - \mu r_1^2| \leq \max\{1,|\mu|\}$, we get

$$|a_3 - \zeta a_2^2| \leq \frac{|e_1|}{\Delta}\max\left\{1, \left|\frac{e_2}{e_1} + \frac{(\zeta\Delta-\varphi)e_1}{\Omega^2}\right|\right\}.$$

In the same way, from (21) and from (22), we get

$$a_3 - \zeta a_2^2 = -\frac{e_1}{\Delta}\left(s_2 + \left(\frac{e_2}{e_1} - \frac{(2\Delta+\varphi-\zeta\Delta)e_1}{\Omega^2}\right)s_1^2\right)$$

and on using $|s_2 - \mu s_1^2| \leq \max\{1, |\mu|\}$, we get

$$|a_3 - \zeta a_2^2| \leq \frac{|e_1|}{\Delta} \max\left\{1, \left|\frac{e_2}{e_1} - \frac{(2\Delta + \varphi - \zeta\Delta)e_1}{\Omega^2}\right|\right\}.$$

□

Corollary 3. *If $f \in \Sigma$ of the style (1) be an element of $\mathcal{W}_\Sigma(\eta, \delta, \lambda, \sigma, \theta, \alpha, \beta, p, q; H_\gamma(\tau, z))$, then*

$$\begin{aligned}
&\left|a_3 - \zeta a_2^2\right| \\
\leq\ & \frac{|1+\gamma-\tau|}{\Delta} \min\left\{\max\left\{1, \left|\frac{\frac{\tau^2}{2} - (\gamma+2)\tau + \frac{(\gamma+1)(\gamma+2)}{2}}{1+\gamma-\tau} + \frac{(\zeta\Delta - \varphi)(1+\gamma-\tau)}{\Omega^2}\right|\right\},\right. \\
& \left.\max\left\{1, \left|\frac{\frac{\tau^2}{2} - (\gamma+2)\tau + \frac{(\gamma+1)(\gamma+2)}{2}}{1+\gamma-\tau} - \frac{(2\Delta + \varphi - \zeta\Delta)(1+\gamma-\tau)}{\Omega^2}\right|\right\}\right\},
\end{aligned}$$

for each $\zeta, \eta, \lambda, \delta$ such that $\zeta \in \mathbb{R}$, $0 \leq \eta \leq 1$, $0 \leq \lambda \leq 1$ and $0 \leq \delta \leq 1$, where Ω, Δ, φ are given by (6) and $H_\gamma(\tau, z)$ is presented by (3).

Theorem 4. *If $f \in \Sigma$ of the style (1) is in the family $\mathcal{K}_\Sigma(\xi, \rho, \sigma, \theta, \alpha, \beta, p, q; h)$, then*

$$\left|a_3 - \zeta a_2^2\right| \leq \frac{|e_1|}{\Phi} \min\left\{\max\left\{1, \left|\frac{e_2}{e_1} + \frac{(\zeta\Phi - \chi)e_1}{\Upsilon^2}\right|\right\}, \max\left\{1, \left|\frac{e_2}{e_1} - \frac{(2\Phi + \chi - \zeta\Phi)e_1}{\Upsilon^2}\right|\right\}\right\},$$

for all ζ, ξ, ρ such that $\zeta \in \mathbb{R}$, $0 \leq \xi \leq 1$ and $0 \leq \rho < 1$, where Υ, Φ, χ are given by (25) and e_1, e_2, a_2 and a_3 as defined in Theorem 2.

Proof. We implement the impressions from the Theorem 2's proof. From (34) and from (35), we get

$$a_3 - \zeta a_2^2 = \frac{e_1}{\Phi}\left(r_2 + \left(\frac{e_2}{e_1} + \frac{(\zeta\Phi - \chi)e_1}{\Upsilon^2}\right)r_1^2\right)$$

by using the certain result $|r_2 - \mu r_1^2| \leq \max\{1, |\mu|\}$, we get

$$|a_3 - \zeta a_2^2| \leq \frac{|e_1|}{\Phi} \max\left\{1, \left|\frac{e_2}{e_1} + \frac{(\zeta\Phi - \chi)e_1}{\Upsilon^2}\right|\right\}.$$

In the same way, from (38) and from (39), we get

$$a_3 - \zeta a_2^2 = -\frac{e_1}{\Phi}\left(s_2 + \left(\frac{e_2}{e_1} - \frac{(2\Phi + \chi - \zeta\Phi)e_1}{\Upsilon^2}\right)s_1^2\right)$$

and on using $|s_2 - \mu s_1^2| \leq \max\{1, |\mu|\}$, we get

$$|a_3 - \zeta a_2^2| \leq \frac{|e_1|}{\Phi} \max\left\{1, \left|\frac{e_2}{e_1} - \frac{(2\Phi + \chi - \zeta\Phi)e_1}{\Upsilon^2}\right|\right\}.$$

□

Corollary 4. *If $f \in \Sigma$ of the style (1) be an element of $\mathcal{K}_{\Sigma}(\xi,\rho,\sigma,\theta,\alpha,\beta,p,q;H_{\gamma}(\tau,z))$, then*

$$\begin{aligned}&\left|a_3-\zeta a_2^2\right|\\ \le\ &\frac{|1+\gamma-\tau|}{\Phi}\min\left\{\max\left\{1,\left|\frac{\frac{\tau^2}{2}-(\gamma+2)\tau+\frac{(\gamma+1)(\gamma+2)}{2}}{1+\gamma-\tau}+\frac{(\zeta\Phi-\chi)(1+\gamma-\tau)}{\Upsilon^2}\right|\right\},\right.\\ &\left.\max\left\{1,\left|\frac{\frac{\tau^2}{2}-(\gamma+2)\tau+\frac{(\gamma+1)(\gamma+2)}{2}}{1+\gamma-\tau}-\frac{(2\Phi+\chi-\zeta\Phi)(1+\gamma-\tau)}{\Upsilon^2}\right|\right\}\right\},\end{aligned}$$

for each ζ,ξ,ρ such that $\zeta \in \mathbb{R}$, $0 \le \xi \le 1$ and $0 \le \rho < 1$, where Υ,Φ,χ are given by (25) and $H_{\gamma}(\tau,z)$ is presented by (3).

3. Conclusions

The main aim of this study was to constitute a new classes $\mathcal{W}_{\Sigma}(\eta,\delta,\lambda,\sigma,\theta,\alpha,\beta,p,q;h)$ and $\mathcal{K}_{\Sigma}(\xi,\rho,\sigma,\theta,\alpha,\beta,p,q;h)$ of bi-univalent functions described through $(p-q)$-Wanas operator and also utilization of the generator function for Laguerre polynomial $L_n^{\gamma}(\tau)$, presented by the equalities in (4) and the producing function $H_{\gamma}(\tau,z)$ given by (3). The initial Taylor-Maclaurin coefficient estimates for functions of these freshly presented bi-univalent function classes $\mathcal{W}_{\Sigma}(\eta,\delta,\lambda,\sigma,\theta,\alpha,\beta,p,q;h)$ and $\mathcal{K}_{\Sigma}(\xi,\rho,\sigma,\theta,\alpha,\beta,p,q;h)$ were produced and the well-known Fekete-Szegö inequalities were examined.

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