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Research Article

On Necessary Condition for the Variable Exponent Hardy Inequality

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We derive a necessary condition for exponent functions p, β such that the variable exponent Hardy inequality $\|x^{\beta(x)-1}\int_0^x f(t)dt\|_{L^{p(\cdot)}(0,l)} \le C\|x^{\beta(x)} f\|_{L^{p(\cdot)}(0,l)}$ holds.

1. Introduction

A sufficient condition on measurable functions $p : (0,l) \to [1,\infty)$, $\beta : (0,l) \to (-\infty,\infty)$ for which the variable exponent Hardy inequality

$$\left\| |x|^{\beta(\cdot)-1} Hf \right\|_{L^{p(\cdot)}(0,l)} \le C \left\| |x|^{\beta(\cdot)} f \right\|_{L^{p(\cdot)}(0,l)}, \quad Hf(x) = \int_0^x f(t)dt \tag{1.1}$$

holds for all $f \ge 0$ have been known (see [1–3]). According to mentioned works, if $\beta(0) < 1 - (1/p(0))$, $p^- := \inf\{p(x) : x \in (0,l)\} > 1$, then a sufficient condition is $p, \beta \in \Lambda$, where Λ means a class of measurable functions $g : (0,l) \to (-\infty,\infty)$ such that $\exists g(0), C > 0$

$$\left| g(x) - g(0) \right| \ln \frac{1}{|x|} \le C, \quad 0 < x < \frac{1}{2}. \tag{1.2}$$

The purpose of this paper is to prove that a weaker continuity condition on $p(\cdot)$ and $\beta(\cdot)$ is necessary for the norm inequality to hold provided that $p(\cdot)$ and $\beta(\cdot)$ are monotone (see Theorems 2.1 and 2.2 for a precise statement), which is the following condition:

$$\left|g(2x) - g(x)\right| \ln \frac{1}{x} \leq C, \qquad 0 < x < \frac{1}{2}. \tag{1.3}$$

Note that condition (1.3) is strictly weaker than (1.2). For example, it is satisfied by $p(x) = C/|\log(x)|^{1/2}$. This condition is new and somewhat surprising. Since in the corresponding theorem for the maximal operator, it is known that $p(\cdot)$ need not be continuous, and the problem of determining which exponent conditions are necessary and/or sufficient is an open one.

If the powers are not monotone, it follows from the results of the paper [2] that condition (1.2) is close to be sharp. Also in [2], the necessity of conditions $p^- > 1$ and $\beta(0) < 1 - 1/p(0)$ was proved. Recently, there have been quite a number of papers discussing the Hardy inequality in norms of the variable exponent Lebesgue spaces [3–11].

For problems of boundedness of classical integral operators in variable exponent Lebesgue spaces and regularity results for nonlinear equations with nonstandard growth condition, see monograph [12] and references therein.

2. Main Results

As to the basic properties of spaces $L^{p(\cdot)}$, we refer to [13]. Throughout this paper it is assumed that $p(x)$ is a measurable function in $(0,l)$ taking its values from the interval $[1,\infty)$ with $p^+ := \sup\{p(\mathrm{x}) : x \in (0,l)\} < \infty$. The space of functions $L^{p(\cdot)}(0,l)$ is introduced as the class of measurable functions $f(x)$ in $(0,l)$, which have a finite $I_p(f) := \int_0^l |f(x)|^{p(x)}dx$-modular. A norm in $L^{p(\cdot)}(0,l)$ is given in the form

$$\|f\|_{L^{p(\cdot)}(0,l)} = \inf\left\{\lambda > 0 : I_p\left(\frac{f}{\lambda}\right) \leq 1\right\}. \tag{2.1}$$

There exists a relation between modular and norm, which is expressed by the following inequalities:

$$\|f\|^{p^+}_{L^{p(\cdot)}(0,l)} \leq I_p(f) \leq \|f\|^{p^-}_{L^{p(\cdot)}(0,l)}, \qquad 1 \geq \|f\|_{p(\cdot)}, \tag{2.2}$$

$$\|f\|^{p^-}_{L^{p(\cdot)}(0,l)} \leq I_p(f) \leq \|f\|^{p^+}_{L^{p(\cdot)}(0,l)}, \qquad 1 \leq \|f\|_{p(\cdot)}. \tag{2.3}$$

Such estimates allow us to perform our estimates in terms of a modular. In the following two theorems, we show that if functions p, β are monotone, then condition (1.3) for them is necessary for inequality (1.1) to hold.

Theorem 2.1. *Let $\beta \in \mathbb{R}$ a function $p : (0,l) \to [1,\infty)$ be increasing on $(0,\varepsilon)$ and such that $p(0) = \lim_{x\to 0} p(x)$ exists, $\beta(0) < 1 - 1/p(0)$, $p^- > 1$. Then for inequality* (1.1) *to hold, it is necessary that for the function $p(\cdot)$ condition* (1.3) *is satisfied.*

Theorem 2.2. *Let $p \in \mathbb{R}$, let $\beta : (0,l) \to [-\infty,\infty)$ be a function decreasing on $(0,\varepsilon)$ such that $\beta(0) = \lim_{x\to 0}\beta(x)$ exists, and let the conditions $\beta(0) < 1 - 1/p(0)$, $p^- > 1$ be satisfied. Then for inequality* (1.1) *to hold, it is necessary that for the function $\beta(\cdot)$ condition* (1.3) *is satisfied.*

The following two theorems show that the logarithmic regularity conditions (1.2) for the functions p, β are essential for inequality (1.1) to hold.

Theorem 2.3. *Let $\beta \in \mathbb{R}$, and $\delta_n = 4^{-n}, n \in \mathbb{N}$. There exist a sequence of functions $\{f_n\}$ and a function $p : (0,l) \to [1,\infty)$ satisfying the conditions $\beta < 1 - 1/p(0)$, $p^- > 1$ such that*

$$\lim_{n\to\infty} \left|p(\delta_n) - p(0)\right| \ln \frac{1}{\delta_n} = \infty, \tag{2.4}$$

and inequality (1.1) *is violated.*

Theorem 2.4. *Let $p > 1$, $\delta_n = 4^{-n}, n \in \mathbb{N}$. Then there exist a sequence of functions $\{f_n\}$ and a function $\beta : (0,l) \to (-\infty,\infty)$ satisfying the conditions $\beta(0) < 1 - 1/p$,*

$$\lim_{n\to\infty} \left|\beta(\delta_n) - \beta(0)\right| \ln \frac{1}{\delta_n} = \infty, \tag{2.5}$$

such that inequality (1.1) *is violated.*

3. Proofs of Main Results

Proof of Theorem 2.1. Denote $I_{p(\cdot)}(f) = \int_0^l |f(t)|^{p(t)} dt$. By (2.2) note that the condition $I_p(f) \le 1$ is equivalent to $\|f\|_{L^{p(\cdot)}(0,l)} \le 1$.

Put $\delta_k = \varepsilon 4^{-k}, k \in \mathbb{N}$, and $f_k(x) = x^{-1/p(x)\,-\beta}\chi_{(\delta_k, 2\delta_k)}(x), x \in (0,l)$. Then for sufficiently large k,

$$\begin{aligned} I_{p(\cdot)}\left(x^{\beta(x)} f_k(x)\right) &= \int_{\delta_k}^{2\delta_k} \left(t^{\beta} t^{-1/p(t)-\beta}\right)^{p(t)} dt \\ &= \int_{\delta_k}^{2\delta_k} t^{-1} dt = \ln 2. \end{aligned} \tag{3.1}$$

Also

$$\begin{aligned} I_{p(\cdot)}\left(x^{\beta(x)-1} H(f_k(x))\right) &\ge \int_{3\delta_k}^{4\delta_k} \left(x^{(\beta-1)} \int_{\delta_k}^{2\delta_k} t^{-(1/p(t))-\beta} dt\right)^{p(x)} dx \\ &\ge C \int_{3\delta_k}^{4\delta_k} \delta_k^{(1-1/p(2\delta_k)-\beta)} x^{(\beta-1)p(2\delta_k)} dx \\ &\ge C\delta_k^{1-p(3\delta_k)/p(2\delta_k)} = Ce^{(1/p^+)[p(3\delta_k)-p(2\delta_k)]\ln(1/2\delta_k)}. \end{aligned} \tag{3.2}$$

Applying inequality (1.1), we have

$$\left|p(3\delta_k)-p(2\delta_k)\right| \ln \frac{1}{2\delta_k} \le C, \qquad k\in\mathbb{N}, \tag{3.3}$$

which by using of monotony of p and its boundedness implies (1.3).

This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. Put $\delta_k = \varepsilon 4^{-k}, k\in\mathbb{N}$ and $f_k(x) = x^{-1/p-\beta(x)}\chi_{(\delta_k,2\delta_k)}(x), x\in(0,l)$. Then

$$\begin{aligned} I_{p(\cdot)}\left(x^{\beta(x)} f_k(x)\right) &= \int_{\delta_k}^{2\delta_k} \left(t^{\beta(t)} t^{-1/p-\beta(t)}\right)^{p(t)} dt \\ &= \int_{\delta_k}^{2\delta_k} t^{-1} dt = \ln 2. \end{aligned} \tag{3.4}$$

Also

$$\begin{aligned} I_{p(\cdot)}\left(x^{\beta(x)-1} H(f_k(x))\right) &\ge \int_{3\delta_k}^{4\delta_k} \left(x^{\beta(x)-1}\int_{\delta_k}^{2\delta_k} t^{-1/p-\beta(t)} dt\right)^{p(x)} dx \\ &\ge C\delta_k^{[\beta(3\delta_k)-\beta(2\delta_k)]p} \ge Ce^{p[\beta(3\delta_k)-\beta(2\delta_k)]\ln(1/\delta_k)}. \end{aligned} \tag{3.5}$$

Applying inequality (1.1), we have

$$\left|\beta(2\delta_k)-\beta(3\delta_k)\right| \ln \frac{1}{\delta_k} \le C, \qquad k\in\mathbb{N} \tag{3.6}$$

which by using monotony of β implies (1.3).

This completes the proof of Theorem 2.2. □

Proof of Theorem 2.3. Let us assume that $f_k(x) = x^{-1/p(x)-\beta}\chi_{(\delta_k,2\delta_k)}(x)$, $x\in(0,l)$. Fix $k\in\mathbb{N}$. We define the step function

$$p(x) = \begin{cases} p_0+\alpha_n & \text{if } x\in(2\delta_n,4\delta_n), \\ p_0 & \text{if } x\in(\delta_n,2\delta_n), \qquad n\in\mathbb{N}. \end{cases} \tag{3.7}$$

Here $\{\alpha_n\}$ is a sequence of positive numbers that satisfies the condition

$$n\alpha_n \longrightarrow \infty \text{ as } n\longrightarrow\infty. \tag{3.8}$$

Then $\alpha_n \ln(1/\delta_n) \to \infty$ as $n \to \infty$; that is, condition (1.2) is not satisfied for the function $p(x)$. Also note that this function $p(\cdot)$ is not monotone. We have

$$I_{p(\cdot)}\left(x^{\beta} f_k(x)\right) = \int_{\delta_k}^{2\delta_k} \left(t^{\beta} t^{-1/p(t)-\beta}\right)^{p(t)} dt = \int_{\delta_k}^{2\delta_k} t^{-1} dt = \ln 2, \tag{3.9}$$

$$\begin{aligned} I_{p(\cdot)}\left(x^{\beta-1} H\left(f_k(x)\right)\right) &\geq \int_{2\delta_k}^{4\delta_k} \left(\int_{\delta_k}^{2\delta_k} t^{-1/p(t)-\beta} dt\right)^{(p_0+\alpha_k)} x^{(\beta-1)(p_0+\alpha_k)} dx \\ &\geq C \int_{2\delta_k}^{4\delta_k} \delta_k^{(1-1/p_0-\beta)(p_0+\alpha_k)} x^{(\beta-1)(p_0+\alpha_k)} dx \\ &\geq C\delta_k^{-\alpha_k/p_0} = Ce^{\alpha_k/p_0 \ln(1/\delta_k)} \longrightarrow \infty \text{ as } k \longrightarrow \infty. \end{aligned} \tag{3.10}$$

The last relation shows violating of inequality (1.1) for sufficiently large k. □

Proof of Theorem 2.4. Let us assume that $f_k(x) = x^{-1/p-\beta(x)} \chi_{(\delta_n, 2\delta_n)}(x)$, $x \in (0, l)$, $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. We define the step function β as

$$\beta(x) = \begin{cases} \beta_0 + \alpha_n & \text{if } x \in (\delta_n, 2\delta_n), \\ \beta_0 & \text{if } x \in (2\delta_n, 4\delta_n), \quad n \in \mathbb{N}, \end{cases} \tag{3.11}$$

where $\alpha_n \ln(1/\delta_n) \to \infty$; that is, condition (1.2) is not satisfied for the function β. Note that this function β is not monotone.

We have

$$I_{p(\cdot)}\left(x^{\beta(x)-1} f_n(x)\right) \geq C\delta_n^{-p\alpha_n} \longrightarrow \infty \text{ as } n \longrightarrow \infty, \tag{3.12}$$

$$I_{p(\cdot)}\left(x^{\beta(x)} f_n(x)\right) \leq C_0 \ln 2. \tag{3.13}$$

The last relation contradicts to inequality (1.1) for sufficiently large k. □

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